ON THE MINIMAL COLORING NUMBER OF EVEN-PARALLELS OF LINKS

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(Accepted November 9, 2018)

Abstract. The minimal coloring number of a \(\mathbb{Z}\)-colorable link is defined as the minimal number of colors for non-trivial \(\mathbb{Z}\)-colorings on diagrams of the link. In this paper, we consider the link obtained by replacing each component of a given link with several parallel strands, which we call a parallel of a link. We show that an even parallel of a link, that is, a parallel of a link with even number of components, is \(\mathbb{Z}\)-colorable except for the case of 2 parallels with non-zero linking number. We then give a simple way to obtain a non-trivial \(\mathbb{Z}\)-coloring for such a parallel of a link with only four colors. This attains the minimal coloring number for such even parallels of links.

1. Introduction

Let \(L\) be a link in the 3-sphere \(S^3\) and \(D\) a regular diagram of \(L\). If a map \(\gamma : \{\text{arcs of } D\} \rightarrow \mathbb{Z}\) satisfies the condition \(2\gamma(a) = \gamma(b) + \gamma(c)\) at each crossing of \(D\) with the over arc \(a\) and the under arcs \(b\) and \(c\), then \(\gamma\) is called a \(\mathbb{Z}\)-coloring on \(D\). A \(\mathbb{Z}\)-coloring which assigns the same integer to all the arcs of the diagram is called a trivial \(\mathbb{Z}\)-coloring. A link is called \(\mathbb{Z}\)-colorable if it has a diagram admitting a non-trivial \(\mathbb{Z}\)-coloring. As usual, we often call the integers of the image of a \(\mathbb{Z}\)-coloring colors.

Let us consider the number of the colors for a non-trivial \(\mathbb{Z}\)-coloring on a diagram of a \(\mathbb{Z}\)-colorable link \(L\). We call the minimum of such number of colors for all non-trivial \(\mathbb{Z}\)-colorings on diagrams of \(L\) the minimal coloring number of \(L\), and denote it by \(\text{mincol}_\mathbb{Z}(L)\).

It is known that the minimal coloring number of any non-splittable \(\mathbb{Z}\)-colorable link is equal to 4 by [4] and [5]. In each of the proof of the result, a procedure was given to obtain a diagram with a \(\mathbb{Z}\)-coloring of 4 colors from any given diagram with a non-trivial \(\mathbb{Z}\)-coloring of a non-splittable \(\mathbb{Z}\)-colorable link. However, from a given diagram of a \(\mathbb{Z}\)-colorable link, by using the procedure, the obtained diagram and \(\mathbb{Z}\)-coloring often becomes very complicated.

In this paper, we give a simple way to obtain a diagram which attains the minimal coloring number for a particular family of \(\mathbb{Z}\)-colorable links. In fact, we consider the link obtained by replacing each component of a given link with several parallel strands, which we call a parallel of a link, defined as follows.

Definition 1.1. Let \(L = K_1 \cup \cdots \cup K_c\) be a link with \(c\) components and \(D\) a diagram of \(L\). For a set \((n_1, \ldots, n_c)\) of integers \(n_i \geq 1\), we denote by \(D^{(n_1, \ldots, n_c)}\) the diagram obtained by taking \(n_i\)-parallel copies of the \(i\)-th component \(K_i\) of \(D\).

Date: December 7, 2018.

2010 Mathematics Subject Classification. 57M25.

Key words and phrases. \(\mathbb{Z}\)-coloring, minimal coloring number.
on the 2-sphere for $1 \leq i \leq c$. The link $L^{(n_1, \ldots, n_c)}$ represented by $D^{(n_1, \ldots, n_c)}$ is called the $(n_1, \ldots, n_c)$-parallel of $L$. When $L$ is a knot, that is $c = 1$, we will say $n$-parallel $L^n$ instead of $(n)$-parallel $L^{(n)}$.

To state our results, for a 2-parallel of a knot, we prepare the following.

**Definition 1.2.** We say a 2-parallel of a knot is *untwisted* if the linking number of the 2 components of the parallel is equal to 0.

Examples of $(n_1, \ldots, n_c)$-parallels of links are shown in Figure 1 and Figure 2.

**Figure 1.** A (3, 2)-parallel of the Hopf link

**Figure 2.** A 2-parallel of the trefoil

We show that an even parallel of a link, that is, a parallel of a link with even number of components, is $\mathbb{Z}$-colorable except for the case of 2-parallels with non-zero linking number.

**Theorem 1.3.** [1] For a non-trivial knot $K$ and any diagram $D$ of $K$, $D^2$ always represents a $\mathbb{Z}$-colorable link if it is untwisted. Moreover, there exists a diagram $D_0$ of $K$ such that $D_0^2$ is locally equivalent to a minimally $\mathbb{Z}$-colorable diagram.

[2] Let $L$ be a non-splittable $c$-component link and $D$ any diagram of $L$. For even numbers $n_1, \ldots, n_c$ at least 4, $D^{(n_1, \ldots, n_c)}$ always represents a $\mathbb{Z}$-colorable link and is locally equivalent to a minimally $\mathbb{Z}$-colorable diagram.

Here we give the definitions used in the theorem above.

**Definition 1.4.** Let $L$ be a $\mathbb{Z}$-colorable link. A diagram $D$ of $L$ is called a *minimally $\mathbb{Z}$-colorable diagram* if there exists a $\mathbb{Z}$-coloring on $D$ which attains the minimal coloring number of $L$.

**Definition 1.5.** For diagrams $D$ and $D'$ of $L$, $D$ is *locally equivalent* to $D'$ if there exist mutually disjoint open subsets $U_1, U_2, \ldots, U_n$ on the 2-sphere such that $D'$ is obtained from $D$ by Reidemeister moves only in $\bigcup_{i=1}^m U_i$. 

(2)
2. Proofs

To prove Theorem 1.3 [1], we prepare the next lemma about the linking number of components of 2-parallel of a knot.

**Lemma 2.1.** Let $D$ a diagram of an oriented knot $K$. For the oriented 2-parallel $K^2 = K_1 \cup K_2$ represented by $D^2$, the linking number of $K_1$ and $K_2$ is equal to the writhe of $D$.

Here we recall that the writhe of $D$ is the sum of the signs of crossings on $D$.

**Proof.** Any crossing $c$ on $D$ is replaced by four crossings by taking 2-parallel copies. In the four crossings, the two crossings consist of the arcs of only $K_1$ or $K_2$, and the other two crossings both $K_1$ and $K_2$.

![Figure 3](image)

Then $c$ and the crossings constructed by the arcs of different components must have same sign. See Figure 3 for the case of a positive crossing. Therefore the linking number of $K_1$ and $K_2$ is equal to the writhe of $D$. \[\square\]

For non-splittability of parallels of knots and links, we can also show the next.

**Lemma 2.2.** [1] Any $n$-parallel of a non-trivial knot is non-splittable for $n \geq 2$.

[2] Any $(n_1, \cdots, n_e)$-parallel of a non-splittable link is non-splittable for $n \geq 1$.

**Proof.** [1] Let $K^n$ be an $n$-parallel of a non-trivial knot $K$. Suppose that $K^n$ is splittable, that is, there exists a 2-sphere $S$ in $S^3 - K^n$ such that $S$ does not bound any 3-ball in $S^3 - K^n$. On the other hand, there exists an embedded annulus $A$ in $S^3$ such that $K^n \subset A$ and the core of $A$ is parallel to the components of $K^n$.

By isotopy of $S$ in $\mathbb{R}^3 - K^n$, let $S \cap A$ be minimized. If $S \cap A$ is the empty set then it is in contradiction with the definition of $S$ and the fact that any knot complement in $S^3$ is irreducible.

We suppose otherwise, that is, $S \cap A$ is not the empty set. We consider a component $C$ of $S \cap A$ on $A$. Then there are two possibilities; (1) $C$ is trivial on $A$, that is, $C$ bounds a disk on $A$, or (2) $C$ is parallel to a component of $K^n$.

In the case (1), $C$ can be removed by isotopy of $S$, since the complement of $A$ is homeomorphic to that of $K$ which is irreducible. That is in contradiction with that $S \cap A$ minimized. In the case (2), since $C$ bounds a disk in $S$, $K$ must be trivial. This also gives a contradiction.

Therefore we conclude that $K^n$ is non-splittable.

[2] From a non-splittable link $L = K_1 \cup \cdots \cup K_e$ with $c$ at least 2, we obtain a parallel $L^{(n_1, \cdots, n_e)} = K_1^{n_1} \cup \cdots \cup K_e^{n_e}$.

We assume $L^{(n_1, \cdots, n_e)}$ is splittable. Then there exists a 2-sphere in $S^3 - L^{(n_1, \cdots, n_e)}$ such that $S^3 = B_1 \cup B_2$ and $L^{(n_1, \cdots, n_e)} = L_1 \cup L_2$ with a link $L_i \subset B_i$ for $i = 1, 2$. Here we take a component $l_i$ from $L_1$ and $l_2$ from $L_2$. 

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In the case that $l_1 \subset K_i^{n_i}$ and $l_2 \subset K_j^{n_j}$ with $i \neq j$. Take $l_k \subset K_k^{n_k}$ for each $k \neq i, j$. Then we see $L' = (l_1 \cup l_2) \cup \bigcup_{k \neq i, j} l_k$ is equivalent to $L$. Now $S$ splits $l_1$ and $l_2$. Since $L'$ is ambient isotopic to $L$, this is contradictory with the assumption.

In the case that $l_1, l_2 \subset K_i^{n_i}$. Take $l_k \subset K_k^{n_k}$ for each $k \neq i$. By $l_1 \subset B_1$, together with the assumption that $L$ is non-splittable, the link $l_1 \cup \bigcup_{k \neq i} l_k$ is contained in $B_1$. On the other hand, for $l_2$, we have $l_2 \cup \bigcup_{k \neq i} l_k$ is contained in $B_2$. This is contradictory to each other. Therefore $L^{(n_1, \cdots, n_c)}$ is non-splittable. □

Proof of Theorem 1.3. [1] Let $K$ be a non-trivial knot in $S^3$. First we give an orientation to the knot $K$. Let us take a diagram $D$ of $K$. Consider the 2-parallel $K^2$ obtained from $D$. Then give the orientation to the 2-parallel $K^2 = K_1 \cup K_2$ induced from that of $K$. Suppose that $K^2$ is untwisted, i.e., the linking number of $K_1$ and $K_2$ is 0. Then we see the writhe of $D$ is 0 by Lemma 2.1. Note that the number of the arcs of $D$ with the positive crossings in both ends is equal to the same number of arcs on $D$ with the negative crossings in both ends as shown in Figure 4. We consider the parallel arcs on $D^2$ corresponding to the arcs mentioned above.

![Figure 4](image)

Here we add a full-twist to the parallel arcs with the sign as shown in Figure 5 and Figure 6.

![Figure 5](image)

![Figure 6](image)

Since the writhe of $D$ is 0, there exist positive crossings as many as negative crossings on $D$. Therefore the diagram obtained by this modification represents the same link $K^2$ as for $D^2$. In the following, we will use the same notation $D^2$ to denote the modified diagram for convenience.

We will show that this $D^2$ admits a non-trivial $\mathbb{Z}$-coloring.
We take a pair of parallel arcs on $D^2$ and set the colors (integers) $a$ and $a + d$ to the pair of parallel arcs on $D^2$, that is, $d$ is a difference of colors of two parallel arcs. We give the colors to remaining arcs around the arcs to satisfy the condition of $\mathbb{Z}$-coloring. See Figure 7. It then follows that $d$ stays constant after passing under adjacent two parallel arcs.

In the same way, the successive parallel arcs can be colored with only crossings shown in Figures 8.

We take a pair of parallel arcs corresponding to the arc on $D$ with the negative crossings in both ends, and fix the colors of two parallel arcs 0 and 1. For the arc colored by 0 is changed to be colored by 2 after passing under another two parallel arcs. See the colors as shown in Figure 9. Thus, the colors $(0, 1)$ and $(2, 3)$ appear alternately by tracing the parallel arcs along $D^2$. We therefore obtain $y = 0, 2$ and $y' = 1, 3$. We see that $2y = 0$ or 4, $2y-1 = -1$ or 3, $2y'-2 = 0$ or 4 and $2y'-3 = -1$ or 3. It follows that $D^2$ admits a $\mathbb{Z}$-coloring $C$ such that $\text{Im}(C) = \{-1, 0, 1, 2, 3, 4\}$. Therefore $K^2$ is $\mathbb{Z}$-colorable.
Here we focus on the arc colored by 4 and −1. We can get a $\mathbb{Z}$-coloring without the colors 4 and −1 by changing the diagram and the coloring obtained above as shown in Figure 10 and Figure 11.

![Figure 10](image10.png)

![Figure 11](image11.png)

We see that $K^2$ also admits the coloring $C'$ such that $\text{Im}(C') = \{0, 1, 2, 3\}$. By Lemma 2.2, $K^2$ is non-splittable. By [4] and [5], the obtained diagram is a minimally $\mathbb{Z}$-colorable diagram, which is locally equivalent to $D^2$.

[2] Let $D^{(n_1, \ldots, n_c)}$ be a diagram of $L^{(n_1, \ldots, n_c)}$. The diagram $D^{(n_1, \ldots, n_c)}$ has only crossings shown in Figure 12.

![Figure 12](image12.png)

We put a circle as fencing the crossings as shown in Figure 13.
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Note that each crossing of $D^{(\ldots, n_k)}$ is contained in some of the regions encircled.

For any parallel family of arcs $(a_1, \ldots, a_k)$ outside the regions, we fix the colors of $a_{k/2}$ and $a_{k/2+1}$ are 1 and the others are 0 as shown in Figure 14.

For the arcs inside the region, we assign the colors as follows.

In the case $n_j = 4m + 4$ for some integer $m$, we assign the colors $-1, 0, 1,$ and 3 as shown in Figure 15.
Then, at each crossing inside the region, the obtained coloring satisfies the condition of \( Z \)-coloring. Here the colors of the arcs intersecting the circle are compatible to those of the arcs outside.

In the case \( n_j = 4m + 2 \), in the same way, we assign the colors \(-1, 0, 1, 2\) as shown in Figure 16.
Again, at each crossing inside the region, the obtained coloring satisfy the condition of $\mathbb{Z}$-coloring. Here the colors of the arcs intersecting the circle are compatible to those of the arcs outside.

We see that $D^{(n_1,\cdots,n_c)}$ admits a $\mathbb{Z}$-coloring $C$ such that $\text{Im}(C) = \{-1,0,1,2\}$ or $\{-1,0,1,2,3\}$. Therefore $L^{(n_1,\cdots,n_c)}$ is $\mathbb{Z}$-colorable.

Moreover, in the case that the color 3 appears, we can delete the color 3 by changing the diagram and the coloring as shown in Figure 17.

![Figure 17](image_url)

**Figure 17.** Delete the color 3

Therefore there exists the coloring $C'$ such that $\text{Im}(C') = \{-1,0,1,2\}$.

Since we are assuming that $L$ is non-splittable, Lemma 2.2 assures that the parallel is non-splittable. Therefore the obtained diagram is a minimally $\mathbb{Z}$-colorable diagram, which is locally equivalent to $D^{(n_1,\cdots,n_c)}$.

\[\square\]

3. Acknowledgement

I would also like to thank Akio Kawauchi for giving me the motivation that I consider about parallels. I am grateful to Kouki Taniyama for pointing out the necessity of Lemmas 2.2. I am also grateful to Ayumu Inoue, Takuji Nakamura and Shin Satoh for useful discussions and advices. I would also like to Kazuhiro Ichihara for his supports and advices.

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