BOUNDS OF EXPONENTS OF QUASIHOMOGENEOUS POLYNOMIALS WITH FIXED INNER MODALITY

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Abstract. The classification of quasihomogeneous polynomials by inner modality, which introduced by V. I. Arnold in [1], was done for inner modality \( \leq 9 \) (see [18],[11],[4],[14]). Moreover the classification of them of corank = 3 was done for inner modality \( \leq 14 \) in [15]. The essential thing in order to perform the classification is to determine bounds of exponents of terms of quasihomogeneous polynomials with fixed inner modality. In this article, for a given \( \mu \in \mathbb{N} \), we will find out a bound \( \varepsilon_\mu \) for which 'If the inner modality of a quasihomogeneous polynomial \( \leq \mu \), then the exponents of terms of the quasihomogeneous polynomial \( \leq \varepsilon_\mu \).'

1. Introduction

V. I. Arnold introduced the notion of modality for isolated hypersurface singularities and he classified all singularities of modality 0, 1 and 2. He named the singularities with modality = 0, 1 and 2 simple one, unimodal one and bimodal one respectively. Here modality means the moduli of singularities in small perturbations of them (see [5]). It is no doubt that this concept has very important meaning for singularities. This work of him has a great influence in various areas of mathematics. But enormous calculation is necessary for further classification of singularities by modality, and it is difficult to execute any more classification. All singularities( simple, unimodal and bimodal) classified are (semi-)quasihomogeneous. So he introduced the notion of inner modality, which can be calculate algebraically, into quasihomogeneous singularities and classified them of inner modality 0,1 actually. He found out these classification to coincide with classification by modality and he conjectured that modality=inner modality in general. His conjecture...

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was shown affirmatively in [12] for the one-dimensional singularities and in [16] in general.

After V. I. Arnol'd, the classification of quasihomogeneous singularities was developed to inner modality \( \leq 9 \) by E. Yoshinaga, M. Suzuki, J. Sarlabous and A. Rodriguez (see [18],[11],[4] and [14]). In the case corank = 3 the classification was done for inner modality \( \leq 14 \) in [15]. They classified singularities by using the formula that inner modality = arithmetical inner modality for their classifications. This formula is not always true, but fortunately, all quasihomogeneous polynomials classified by them satisfy it. Their classifications have been done up to the limit where the formula is satisfied (see [14], [15]). In order to classify quasihomogeneous polynomials we need to find bounds of exponents of terms of them with fixed inner modality. For further classification we have to find a method depending on this formula and have to estimate exponents of them.

The purpose of this article is to find out a bound \( \varepsilon_\mu \) for which "If the inner modality of a quasihomogeneous polynomial \( \leq \mu \), then the exponents of terms of the quasihomogeneous polynomial \( \leq \varepsilon_\mu \)." Using this result we may classify quasihomogeneous singularities with higher inner modality.

2. Preliminaries

In this section, we explain the terms and the results used in this article. The details of our results will be stated in §3.

A local analytic function \( f : (\mathbb{C}^n, O) \rightarrow (\mathbb{C}, 0) \), that is \( f \in \mathcal{M} \subset \mathbb{C}\{x_1, \ldots, x_n\} \), has an isolated singularity if

\[
\left\{ x \mid \frac{\partial f}{\partial x_1}(x) = \cdots = \frac{\partial f}{\partial x_n}(x) = 0 \right\} = \{0\}
\]

locally, where \( \mathcal{M} \) is the maximal ideal of the local ring \( \mathbb{C}\{x_1, \ldots, x_n\} \). It is well known that a local analytic function with isolated singularity is a polynomial up to a suitable local coordinate transformation (see [6],[17]). Hence we will study "quasihomogeneous" polynomials with isolated singularity at the origin.

For positive rational numbers \( r_1, \ldots, r_n \in \mathbb{Q}^+ \), a monomial

\[
m = x_1^{i_1} \cdots x_n^{i_n} \in \mathbb{C}[x_1, \ldots, x_n] \quad (i_1, \ldots, i_n \in \mathbb{N} \cup \{0\})
\]

has generalized degree \( d \) if \( r_1i_1 + \cdots + r_ni_n = d \) and we denote the generalized degree of \( m \) by \( \text{gdeg}(m) \). A polynomial \( f \in \mathbb{C}[x_1, \ldots, x_n] \) is quasihomogeneous of type \( (d; r_1, \ldots, r_n) \) if each monomial term of \( f \) with non-zero coefficient has generalized degree \( d \). Then we call the number \( d \) the generalized degree of \( f \) and call \( r_i \)'s the weights of \( f \).
A local analytic function \( f : (\mathbb{C}^n, O) \to (\mathbb{C}, 0) \) has a quasihomogeneous singularity at the origin if \( f \) becomes a quasihomogeneous polynomial after a suitable local coordinate transformation.

**Theorem 2.1** ([9]). Suppose that \( f \in \mathcal{M} \subset \mathbb{C}\{x_1, \ldots, x_n\} \) has a quasihomogeneous isolated singularity. Then there exist a coordinate system \((y_1, \ldots, y_n)\) in which \( f \) has the form

\[
f = h(y_1, \ldots, y_k) + y_{k+1}^2 + \cdots + y_n^2
\]

with a quasihomogeneous polynomial \( h \in \mathbb{C}[y_1, \ldots, y_k] \) of type \((1; s_1, \ldots, s_k)\) \((0 < s_i < \frac{1}{2}, i = 1, \ldots, k)\). The natural number \( k \) and \((s_1, \ldots, s_k)\) are uniquely determined up to permutations of components.

We call the number \( k \) the corank of \( f \) and call the polynomial \( h \) the residual part of \( f \). We denote the corank of \( f \) by \( \text{corank}(f) \). In order to classify them, it is sufficient to classify their residual parts.

The following proposition and theorem are frequently used in this article.

**Proposition 2.2** ([10]). Let \( f \) be the residual part of a quasihomogeneous polynomial with isolated singularity at the origin. Then for any \( i (i = 1, \cdots, k) \), there are integers \( m_i \geq 2 \) and \( j_i \) \((j_i = 1, \ldots, k)\) such that \( f \) contains the monomial \( x_i^{m_i} x_j \) as a term.

**Theorem 2.3** ([10]). Let \( f \) be the residual part of a quasihomogeneous polynomial with isolated singularity at the origin and suppose that \( f \) have the type \((1; r_1, \ldots, r_k)\) \((0 < r_i < \frac{1}{2}, i = 1, \ldots, k)\). Then

\[
\sum_{i=1}^{k} r_i \leq \frac{k}{3}.
\]

The inequality in the above theorem is derived from Proposition 2.12 in K. Saito [10]. From this theorem it follows easily that the minimum of \( r_i \) is less than or equal to \( \frac{1}{3} \).

We denote the ideal \( \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right) \) of the ring \( \mathbb{C}\{x_1, \ldots, x_n\} \) by \( \Delta(f) \) and denote the quotient ring \( \mathbb{C}\{x_1, \ldots, x_n\}/\Delta(f) \) by \( \mathcal{R}_f \). Note that by Hilbert’s Nullstellensatz, \( \mathcal{M}^p \subset \Delta(f) \subset \mathbb{C}\{x_1, \ldots, x_n\} \) for some \( p \in \mathbb{N} \) if \( f \) has an isolated singularity at the origin. Then the dimension of \( \mathcal{R}_f \) over \( \mathbb{C} \) is finite.

From now on let \( f \) be a quasihomogeneous polynomial of type \((1; r_1, \ldots, r_n)\) with isolated singularity at the origin. V.I. Arnol’d showed in [1] that the number of basis monomials of \( \mathcal{R}_f \) with given generalized degree (for
given \((r_1, \ldots, r_n)\) is the same for all quasihomogeneous polynomials \(f\) of the same type as follows.

**Theorem 2.4** ([1]). Let \(r_1, \ldots, r_n\) be positive rational numbers for which \(r_i = A_i/N\) \((i = 1, \ldots, n)\), where \(N\) and \(A_i\)’s are positive natural numbers. If \(f\) is a quasihomogeneous polynomial of type \((1; r_1, \ldots, r_n)\) with isolated singularity at the origin, then

\[
\sum \mu_j z^j = \prod_{j=1}^{n} \frac{z^{N-A_j} - 1}{z^{A_j} - 1},
\]

where \(\mu_i\) is the number of basis monomials in \(R_f\) with generalized degree \(i/N\).

We denote the right side of the equality in the above theorem by \(\chi_f(z)\) and we call it the characteristic function of \(f\), and when it becomes a polynomial, it is especially called the characteristic polynomial of \(f\). By the above theorem we can define the following.

**Definition 2.1** ([1]). We call the number of basis monomials of \(R_f\) with generalized degree \(\geq 1\) the inner modality of \(f\) and it is denoted by \(m(f)\).

We see that the highest degree of generalized degrees of basis monomials of \(R_f\) is \(n - 2 \sum r_i\) and it is denoted by \(d_f\). The coefficients of \(\chi_f(z)\) are symmetric because it is the product of cyclotomic polynomials. Hence we have

\[
m(f) = \sum_{j \geq N} \mu_j = \sum_{j \leq D - N} \mu_j,
\]

where \(D = nN - 2 \sum A_j = N(n - 2 \sum r_j) = N d_f\). Hence \(m(f)\) is the number of basis monomials of \(R_f\) with generalized degree \(\leq d_f - 1\).

For a (semi-)quasihomogeneous polynomial \(f\) of type \((1; r_1, \ldots, r_n)\), the following invariant is introduced in [14].

**Definition 2.2.** The number of monomials in \(\mathbb{C}[x_1, \ldots, x_n]\) with generalized degree \(\leq d_f - 1\) is called the arithmetic inner modality of \(f\) and it is denoted by \(m_0(f)\).

By the definition, we have \(m(f) \leq m_0(f)\) in general. If the images of monomials in \(R_f\) with \(gdeg \leq d_f - 1\) are linearly independent, we have \(m(f) = m_0(f)\). In [14] the following result about \(m(f)\) and \(m_0(f)\) is given.

**Proposition 2.5.** We have \(m(f) = m_0(f)\) if and only if

\[
gdeg \left( \frac{\partial f}{\partial x_i} \right) > d_f - 1
\]
for any \( i \) (\( i = 1, \ldots, k \)).

3. Results and Proofs

Let \( f \) be a quasihomogeneous polynomial of type \((1; r_1, \ldots, r_k)\) \((0 < r_i < \frac{1}{2}, i = 1, \ldots, k)\) with isolated singularity at the origin. By Proposition 2.2 \( f \) contains monomials \( x_1^{m_1} x_j^1, \ldots, x_k^{m_k} x_j^k \) for some integers \( m_1, \ldots m_k \) \((m_1, \ldots, m_k \geq 2)\) and some \( j_i \) \((j_i = 1, \ldots, k)\) with non-zero coefficients. Conversely the type of a quasihomogeneous polynomial containing such monomials is uniquely determined. Then for a fixed \( m(f) \) we can determine bounds of the exponents \( m_1, \ldots m_k \).

Theorem 3.1.

\begin{enumerate}
  \item For \( \text{corank}(f) = 2 \), if \( 1 \leq m(f) \leq \mu \), then we have
    \[
    m_1, m_2 < 3(\mu + 2)
    \]
  \item For \( \text{corank}(f) = 3 \), if \( m(f) \leq \mu \), then we have
    \[
    m_1, m_2, m_3 \leq \max\{3(\mu + 2), \mu^2 - 1\}.
    \]
  \item For \( \text{corank}(f) \geq k \geq 4 \), if \( m(f) \leq \mu \), then we have
    \[
    m_1, \ldots m_k \leq \max\left\{\frac{3\mu}{k-3}, (\mu-k)(\mu-k+2)\right\}.
    \]
\end{enumerate}

For \( \text{corank} = 2, 3 \) we use the following propositions in the proofs of our results.

Proposition 3.2 ([2]). For every quasihomogeneous polynomial of corank \( = 2 \) with isolated singularity at the origin, its residual part contains at least one of the three sets of monomials in the following table with a suitable numbering of variables.

<table>
<thead>
<tr>
<th>Class Monomials</th>
<th>Class monomials</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I ) ( x^a, y^b )</td>
<td>( III ) ( x^a y^b x )</td>
</tr>
<tr>
<td>( II ) ( x^a, y^b x )</td>
<td></td>
</tr>
</tbody>
</table>

Proposition 3.3 ([2]). For every quasihomogeneous polynomial of corank \( = 3 \) with isolated singularity at the origin, its residual part contains at least one of the seven sets of monomials in the following table with a suitable numbering of variables.
Proof of (1). Let $f$ be a quasihomogeneous polynomial of type $(1; r_1, r_2)$ and let $r_{\min} := \min\{r_1, r_2\}$. Then $f$ is classified in the three classes I, II, III according to the sets of monomials listed in Proposition 3.2 which $f$ contains with non-zero coefficients. The next table is helpful for this proof.

<table>
<thead>
<tr>
<th>Class</th>
<th>Monomials</th>
<th>Class</th>
<th>Monomials</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$x^a, y^b, z^c$</td>
<td>V</td>
<td>$x^a, y^b z, z^c x$</td>
</tr>
<tr>
<td>II</td>
<td>$x^a, y^b, y z^c$</td>
<td>VI</td>
<td>$x^a y, y^b x, z^c x$</td>
</tr>
<tr>
<td>III</td>
<td>$x^a, y^b x, z^c x$</td>
<td>VII</td>
<td>$x^a y, y^b z, z^c x$</td>
</tr>
<tr>
<td>IV</td>
<td>$x^a, y^b z, z^c y$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note that $r_1 \leq \frac{1}{a}$, $r_2 \leq \frac{1}{b}$ for any class. We divide the proof into two cases.

Case I. The case $\mu r_{\min} > d_f - 1$.

First we consider the case $a, b \geq 3$. Suppose that $r_{\min} = r_1 \leq r_2$ ($a \geq b$). Then by the hypothesis $\mu r_{\min} > d_f - 1 = 1 - 2(r_1 + r_2)$

$$\frac{\mu + 2}{a} \geq (\mu + 2) r_1 > 1 - 2r_2 \geq 1 - 2 \times \frac{1}{3} = \frac{1}{3}.$$

Hence we have $a < 3(\mu + 2)$ and thus $a, b < 3(\mu + 2)$. We have the same result in the case $r_1 \geq r_2 = r_{\min}$.

Next we consider the case $a = 2$ or $b = 2$. In this case the class of $f$ must be II or III. If the class of $f$ is II, $b = 2$ because $a \geq 3$. Then

$$r_1 = \frac{1}{a}, \quad r_2 = \frac{a - 1}{ab},$$

$$d_f - 1 = 1 - 2(r_1 + r_2) = 1 - 2\left(\frac{1}{a} + \frac{a - 1}{2a}\right) = -\frac{1}{a} < 0.$$

This means $m(f) = 0$, which contradicts the hypothesis $1 \leq m(f)$.

Hence the class of $f$ can not be II. If the class of $f$ is III, then $r_1 = \frac{b - 1}{ab - 1}$.
Now suppose that \( a = 2 \). Then

\[
d_f - 1 = 1 - 2(r_1 + r_2) = 1 - 2\left( \frac{b - 1}{2b - 1} + \frac{1}{2b - 1} \right) = -\frac{1}{2b - 1} < 0
\]

This means \( m(f) = 0 \), but it contradicts the hypothesis \( 1 \leq m(f) \). Thus the case \( a = 2 \) is impossible. By the same reason the case \( b = 2 \) is also impossible. Hence the class of \( f \) can not be III.

**Case II.** The case \( \mu r_{\min} \leq d_f - 1 \).

Suppose that \( r_{\min} = r_1 \). Let \( \mathcal{M} \) be the set of \( 1, x_1, x_1^2, \ldots, x_1^m \). If \( \mathcal{M} \) is linearly independent in \( \mathcal{R}_f \) over \( \mathbb{C} \), then we have \( m(f) > \mu \) because that \( \text{gdeg}(1), \text{gdeg}(x_1), \ldots, \text{gdeg}(x_1^m) \leq d_f - 1 \) by the hypothesis \( \mu r_{\min} \leq d_f - 1 \). It contradicts the hypothesis \( m(f) \leq \mu \). Hence there exist \( \lambda_0, \lambda_1, \ldots, \lambda_\mu, \) not all zero, such that

\[
(*) \quad \lambda_0 1 + \lambda_1 x_1 + \cdots + \lambda_\mu x_1^\mu \in \Delta(f).
\]

But for any \( i \) (\( i = 1, 2 \))

\[
\text{gdeg}\left( \frac{\partial f}{\partial x_i} \right) = 1 - r_i > 1 - 2(r_1 + r_2) = d_f - 1.
\]

It contradicts (\( * \)). We have also a contradiction in the case \( r_{\min} = r_2 \) similarly. Hence these cases are impossible.

Therefore we have \( m_1, m_2 < 3(\mu + 2) \) in all cases. \( \square \)

**Proof of (2).** Let \( f \) be quasihomogeneous of type \( (1; r_1, r_2, r_3) \) and let \( r_{\min} := \min\{r_1, r_2, r_3\} \). Then \( f \) is classified in seven classes according to the sets of monomials listed in Proposition 3.3 which \( f \) contains with non-zero coefficients. The next table is helpful for this proof.

<table>
<thead>
<tr>
<th>Class</th>
<th>monomials</th>
<th>( r_1 )</th>
<th>( r_2 )</th>
<th>( r_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>( x^a, y^b, z^c )</td>
<td>( \frac{1}{a} )</td>
<td>( \frac{1}{b} )</td>
<td>( \frac{1}{c} )</td>
</tr>
<tr>
<td>II</td>
<td>( x^a, y^b, yz^c )</td>
<td>( \frac{1}{a} )</td>
<td>( \frac{1}{b} )</td>
<td>( \frac{b-1}{bc} )</td>
</tr>
<tr>
<td>III</td>
<td>( x^a, y^b x, z^c x )</td>
<td>( \frac{1}{a} )</td>
<td>( \frac{a-1}{ab} )</td>
<td>( \frac{a-1}{ac} )</td>
</tr>
<tr>
<td>IV</td>
<td>( x^a, y^b z, z^c y )</td>
<td>( \frac{1}{a} )</td>
<td>( \frac{bc-1}{bc} )</td>
<td>( \frac{b-1}{bc} )</td>
</tr>
<tr>
<td>V</td>
<td>( x^a, y^b z, z^c x )</td>
<td>( \frac{1}{a} )</td>
<td>( \frac{ac-a+1}{abc} )</td>
<td>( \frac{a-1}{ac} )</td>
</tr>
<tr>
<td>VI</td>
<td>( x^a y^b, y^b x, z^c x )</td>
<td>( \frac{b-1}{ab-1} )</td>
<td>( \frac{a-1}{ab-1} )</td>
<td>( \frac{(a-1)b}{(ab-1)c} )</td>
</tr>
<tr>
<td>VII</td>
<td>( x^a y^b z, z^c x )</td>
<td>( \frac{bc-c+1}{abc+1} )</td>
<td>( \frac{ac-a+1}{abc+1} )</td>
<td>( \frac{ab-b+1}{abc+1} )</td>
</tr>
</tbody>
</table>
Note that \( r_1 \leq \frac{1}{a} \), \( r_2 \leq \frac{1}{b} \), \( r_3 \leq \frac{1}{c} \) for any class. We divide the proof into two cases.

**Case I.** The case \( \mu r_{\min} > \frac{d_f - 1}{a} \).

First we consider the case where two of \( a \), \( b \) and \( c \) are greater than or equal to 3. Suppose that \( r_1 = r_{\min} \). Since \( \mu r_1 > 2 - 2(r_1 + r_2 + r_3) \), for any \( i \ (i = 1, 2, 3) \)

\[(\mu + 2)r_i \geq (\mu + 2)r_1 > 2 - 2(r_2 + r_3) \geq 2 - 2\left(\frac{1}{b} + \frac{1}{c}\right) \geq 2 - 2\left(\frac{1}{2} + \frac{1}{3}\right) = \frac{1}{3}.
\]

Hence \( \frac{\mu + 2}{a} \leq \frac{\mu + 2}{b} \leq \frac{\mu + 2}{c} \geq \frac{1}{3} \) and thus \( a, b, c \leq 3(\mu + 2) \). Also in the case where \( r_2 \) or \( r_3 \) is \( r_{\min} \), we obtain the same result similarly.

Next we consider the case where two of \( a \), \( b \) and \( c \) are equal to 2. In this case the class of \( f \) is III, IV, V, VI or VII.

In the case where the class of \( f \) is III, we obtain \( b = c = 2 \) because \( a \geq 3 \). Then \( \frac{1}{a} = r_1 \leq r_2 = r_3 = \frac{a - 1}{2a} \). If \( r_1 < r_2 = r_3 \) then for any non-negative integers \( m, n \ (m + n \geq 3) \)

\[mr_2 + nr_3 = (m + n)r_2 > 2r_2 + r_1 = 1.
\]

It means that every term of \( f \) contains the variable \( x \) and \( f \) is reducible. But it contradicts the assumption that \( f \) has an isolated singularity at the origin. Thus \( r_1 = r_2 = r_3 = \frac{1}{3} \) from \( 2r_2 + r_1 = 2r_3 + r_1 = 1 \). Hence we have \( a = 3 \).

In the case where the class of \( f \) is IV, we obtain \( b = c = 2 \) because \( a \geq 3 \). Then \( \frac{1}{a} = r_1 \leq r_2 = r_3 = \frac{1}{3} \). Thus

\[\frac{\mu + 2}{a} \geq (\mu + 2)r_1 > 2 - 2(r_2 + r_3) \geq 2 - 2\left(\frac{1}{3} + \frac{1}{3}\right) = \frac{2}{3}.
\]

Hence we have \( a < \frac{3(\mu + 2)}{2} \).

In the case where the class of \( f \) is V, we obtain \( b = c = 2 \) because \( a \geq 3 \). Then \( \frac{1}{a} = r_1 \leq r_2 = r_3 = \frac{a + 1}{4a} \). Since \( \mu r_1 > 2 - 2(r_1 + r_2 + r_3) \),

\[\frac{\mu + 2}{a} \geq (\mu + 2)r_1 > 2 - 2(r_2 + r_3) = 2 - 2\times\frac{3a - 1}{4a} > 2 - 2\times\frac{3}{4} = \frac{1}{2}.
\]

Hence \( a < 2(\mu + 2) \).

In the case where the class of \( f \) is VI, first we consider the case \( a = b = 2 \). Then \( \frac{2}{3x} = r_3 \leq r_1 = r_2 = \frac{1}{3} \). Since \( \mu r_3 > 2 - 2(r_1 + r_2 + r_3) \), we have

\[\frac{2(\mu + 2)}{3c} = (\mu + 2)r_3 > 2 - 2(r_1 + r_2) = 2 - 2\times\frac{2}{3} = \frac{2}{3}.
\]

It follows that \( c < \mu + 2 \). Next we consider the case \( b = c = 2 \). Then \( \frac{1}{2a - 1} = r_1 \leq r_2 = r_3 = \frac{a - 1}{2a - 1} \). If \( r_1 < r_2 = r_3 \), then for any non-negative
integer $m, n \ (m + n \geq 3)$

$$mr_2 + nr_3 = (m + n)r_2 > 2r_2 + r_1 = 1.$$  

It means that every term of $f$ contains the variable $x$ and $f$ is reducible. But it contradicts the assumption that $f$ has an isolated singularity at the origin. Hence $r_1 = r_2 = r_3 = \frac{1}{3}$ because $2r_2 + r_1 = 2r_3 + r_1 = 1$. Thus $a = 3$. Finally we consider the case $c = a = 2$ and then $\frac{1}{2b-1} = r_2 \leq r_3 = \frac{b}{2(2b-1)} \leq r_1 = \frac{b-1}{2b-1}$. Since $\mu r_2 > 2 - 2(r_1 + r_2 + r_3)$, we have

$$\frac{\mu + 2}{b} \geq (\mu + 2)r_2 > 2 - 2(r_1 + r_3) = 2 - 2 \times \frac{3b - 2}{4b - 2} > 2 - 2 \times \frac{3}{4} = \frac{1}{2}. $$

Hence $b < 2(\mu + 2)$.

In the case where the class of $f$ is VII, because of the symmetry of $x, y, z$, it is enough to consider the case $b = c = 2$. If $a \geq 3$, $\frac{4}{4a+1} = r_1 \leq r_2 = \frac{a+1}{4a+1} \leq r_3 = \frac{2a-1}{4a+1}$. Since $\mu r_1 > 2 - 2(r_1 + r_2 + r_3)$, we have

$$\frac{\mu + 2}{a} \geq (\mu + 2)r_1 > 2 - 2(r_2 + r_3) = 2 - 2 \times \frac{3a}{4a + 1} > 2 - 2 \times \frac{3}{4} = \frac{1}{2}.$$ 

Hence $a < 2(\mu + 2)$.

**Case II. The case $\mu r_{\min} \leq d_f - 1$.**

First we consider the case $r_{\min} = r_1 \leq r_2 \leq r_3$. Then

$$\text{gdeg}(1), \text{gdeg}(x_1), \ldots, \text{gdeg}(x_1^\mu) \leq d_f - 1.$$ 

Hence if $\mathcal{M} := \{1, x_1, x_1^2, \ldots, x_1^\mu\}$ is linearly independent in $\mathcal{R}_f$ over $\mathbb{C}$ then we have $m(f) > \mu$. But it contradicts the hypothesis and thus $\mathcal{M}$ is linearly dependent. Then there exist some scalars $\lambda_0, \ldots, \lambda_\mu$, not all zero, such that

$$\lambda_0 1 + \lambda_1 x_1 + \ldots + \lambda_\mu x_1^\mu \in \Delta(f).$$

On the other hand, since for any $i \ (i = 1, 2, 3)$

$$\text{gdeg} \left( \frac{\partial f}{\partial x_i} \right) = 1 - r_i \geq \frac{1}{2} > \text{gdeg}(x_1), \text{gdeg}(1) = 0,$$

we have $\lambda_0 = \lambda_1 = 0$. Hence there exists some $l \ (l = 2, \ldots, \mu)$ such that $\lambda_l \neq 0$. Then $f$ contains a term $x_1^l x_i$ with non-zero coefficient for some $i \ (i = 1, 2, 3)$ and thus $lr_1 + r_i = 1$. On the other hand $f$ contains a term $x_1^m x_j$ for some $m \ (m = 2, 3, \ldots)$ and some $j \ (j = 1, 2, 3)$ and thus $mr_3 + r_j = 1$. Then we have $r_3 \geq \frac{1}{l+1}$ because $(l + 1)r_3 \geq lr_1 + r_i = 1$.  

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(9)
Hence for any $i$ ($i = 1, 2, 3$)

$$lr_1 = 1 - r_i \geq 1 - r_3 = (m - 1)r_3 + r_j \geq (m - 1)r_3 + r_1$$

$$(l - 1)r_i \geq (l - 1)r_1 \geq (m - 1)r_3 \geq \frac{m - 1}{l + 1}.$$ 

Hence $a, b, c \leq \frac{(l - 1)(l + 1)}{m - 1} \leq (\mu - 1)(\mu + 1)$.

By similar arguments as above, we have the same result regardless of the order of the weights $r_1, r_2, r_3$.

Therefore we have $m_1, m_2, m_3 \leq (\mu - 1)(\mu + 1)$ for any class. \qed

**Proof of (3).** Let $f$ be quasihomogeneous of type $(1; r_1, \ldots, r_k)$. By Proposition 2.2, for any $i$ ($i = 1, \ldots, k$), there are integers $m_i$ ($\geq 2$) and $j_i$ ($j_i = 1, \ldots, k$) such that $f$ contains the monomial $x_i^{m_i} x_j$ as a term. Then note that $r_1 \leq \frac{1}{m_1}, \ldots, r_k \leq \frac{1}{m_k}$. We divide the proof into two cases. First we consider the case $r_1 \leq \cdots \leq r_k$.

**Case I.** The case $r_i > d_f - 1$ for some $i$ ($i = 1, \ldots, k$).

In this case we have $\frac{1}{d_f} > r_k \geq r_i > d_f - 1$ and $1 - r_k > \frac{1}{d_f} > r_i > d_f - 1$.

Hence we have $m(f) = m_0(f)$ by Proposition 2.5. If $\mu r_1 \leq d_f - 1$ the monomials $x_1^{m_1}, x_1^{m_2}, \ldots, x_1^{m_k}$ have the generalized degrees $\leq d_f - 1$. Then we have $m(f) = m_0(f) > \mu$ and it contradicts the hypothesis $m(f) \leq \mu$. It follows that $\mu r_1 > d_f - 1 = k - 1 - 2(r_1 + \cdots + r_k)$ and for any $j$ ($j = 1, \ldots, k$) we have

$$\mu r_j \geq \mu r_1 > k - 1 - 2(r_1 + \cdots + r_k) \geq k - 1 - 2 \times \frac{k}{3} = \frac{k - 3}{3}$$

by Saito’s inequality $r_1 + \cdots + r_k \leq \frac{k}{3}$ (see Theorem 2.3). Hence we have $\frac{\mu}{m_1} \cdots, \frac{\mu}{m_k} > \frac{k - 3}{3}$ and thus $m_1, \ldots, m_k \leq \frac{3\mu}{k - 3}$.

**Case II.** The case $r_i \leq d_f - 1$ for any $i$ ($i = 1, \ldots, k$).

Moreover we divide this case into two cases.

**Case II-1.** The case $(\mu - k + 1)r_1 > d_f - 1$.

By Saito’s inequality,

$$d_f - 1 = k - 1 - 2(r_1 + \cdots + r_k) \geq k - 1 - 2 \times \frac{k}{3} = \frac{k - 3}{3}.$$ 

Then for any $j$ ($j = 1, \ldots, k$),

$$(\mu - k + 1)r_j \geq (\mu - k + 1)r_1 > d_f - 1 \geq \frac{k - 3}{3}$$

and thus we have $m_1, \ldots, m_k \leq \frac{3(\mu - k + 1)}{k - 3} < \frac{3\mu}{k - 3}$.

**Case II-2.** The case $(\mu - k + 1)r_1 \leq d_f - 1$.

Then the generalized degrees of the monomials $1, x_1, \ldots, x_k, x_1^2, \ldots, x_1^{\mu - k + 1}$
are less than or equal to \(d_f-1\). Hence if \(\mathcal{M} := \{1, x_1, \ldots, x_k, x_1^2, \ldots, x_1^{\mu-k+1}\}\) is linearly independent over \(\mathbb{C}\) in \(\mathcal{R}_f\), then \(m(f) \geq \mu + 1\). But it contradicts the hypothesis \(m(f) \leq \mu\) and thus \(\mathcal{M}\) is linearly dependent. Then there exist some scalars \(\lambda_0, \ldots, \lambda_\mu\), not all zero, such that

\[
\lambda_0 \cdot 1 + \lambda_1 x_1 + \cdots + \lambda_k x_k + \lambda_{k+1} x_1^2 + \cdots + \lambda_\mu x_1^{\mu-k+1} \in \Delta(f).
\]

Since the degree of each term of \(\frac{\partial f}{\partial x_i}\) \((i = 1, \ldots, k)\) is greater than 1, the scalars \(\lambda_0, \ldots, \lambda_k\) must be 0. Hence \(\lambda_{k+1}, \ldots, \lambda_{\mu-1}\) or \(\lambda_\mu\) are not zero and \(f\) contains a monomial \(x_1^l x_i\) with non-zero coefficient for some integer \(l\) \((2 \leq l \leq \mu - k + 1)\) and some \(i\) \((i = 1, \ldots, k)\). Then we have \(lr_1 + r_i = 1\) and thus

\[
(l + 1)r_k \geq lr_1 + r_i = 1, \quad r_k \geq \frac{1}{l + 1}.
\]

On the other hand, by Proposition 2.2 \(f\) contains a monomial \(x_k^m x_j\) with non-zero coefficient for some integer \(m\) \((2 \leq m)\) and some \(j\) \((j = 1, \ldots, k)\) and thus we have \(mr_k + r_j = 1\). Hence we have

\[
lr_1 = 1 - r_i \geq 1 - r_k = (m - 1)r_k + r_j \geq (m - 1)r_k + r_1,
\]

and thus for any \(p\) \((p = 1, \ldots, k)\),

\[
(l - 1)r_p \geq (l - 1)r_1 \geq (m - 1)r_k \geq \frac{m - 1}{l + 1}.
\]

Hence

\[
m_1, \ldots, m_k \leq \frac{(l - 1)(l + 1)}{m - 1} \leq \frac{(\mu - k)(\mu - k + 2)}{2 - 1} = (\mu - k)(\mu - k + 2).
\]

By similar arguments as above, we have the same result regardless of the order of the weights \(r_1, \ldots, r_k\). This completes the proof. \(\square\)

**References**


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