A TRIGONOMETRIC PROOF OF THE STEINER-LEHMUS THEOREM IN HYPERBOLIC GEOMETRY

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Abstract. We give a trigonometric proof of the Steiner-Lehmus Theorem in hyperbolic geometry. Precisely we show that if two internal bisectors of a triangle on the hyperbolic plane are equal, then the triangle is isosceles.

1. Introduction

In 1844 [6], Steiner gave the first proof of the following theorem. If two internal bisectors of a triangle on the Euclidean plane are equal, then the triangle is isosceles. This had been originally asked by Lehmus in 1840, and now is called the Steiner-Lehmus Theorem. Since then, wide variety of proofs have been given by many people over 170 years. At present, at least 80 different proofs exist. See [4]. For example, in 2008, Hajja gave a short trigonometric proof in [3]. On the other hand, several proofs of this theorem in hyperbolic geometry ware given in [5], [1] and [2]. In [5], the theorem is differently stated, and its hyperbolic trigonometric proof would be incorrect. Then, as a part of study of gyrogroups, the theorem was considered in the homonymous paper [1], and also, an alternative proof was given in [2] by using gyrogroups. In this paper, we correct and complete a hyperbolic trigonometric proof based on [3].

2. Steiner-Lehmus Theorem

Theorem. If two internal bisectors of a triangle on the Hyperbolic plane are equal, then the triangle is isosceles.

Proof. We consider a triangle $ABC$ on the hyperbolic plane. See Figure 1. Let $B'$ be the intersection of the side $AC$ and the internal bisector of the angle $B$. Let $C'$ be the intersection of the side $AB$ and the internal bisector of the angle $C$. Then $BB'$ and $CC'$ are the internal bisectors of the angles $B$ and $C$ respectively. Let $a, b$ and $c$ be the lenghths of the opposite sides of the angles $A, B$ and $C$ respectively. We set $\beta = B/2$, $\gamma = C/2$, $u = AB'$, $U = B'C$, $v = AC'$, and $V = C'B$.
We apply the law of sines in hyperbolic geometry to the triangles \(ABC\), \(BCC'\), \(ACC'\), \(CBB'\) and \(ABB'\) respectively, then we have the following.

\[
\begin{align*}
(1) \quad \frac{\sinh a}{\sin A} &= \frac{\sinh b}{\sin 2\beta} = \frac{\sinh c}{\sin 2\gamma}, \\
(2) \quad \frac{\sinh CC'}{\sin 2\beta} &= \frac{\sinh V}{\sin \gamma}, \\
(3) \quad \frac{\sinh CC'}{\sin A} &= \frac{\sinh v}{\sin \gamma}, \\
(4) \quad \frac{\sinh BB'}{\sin 2\gamma} &= \frac{\sinh U}{\sin \beta}, \\
(5) \quad \frac{\sinh BB'}{\sin A} &= \frac{\sinh u}{\sin \beta}.
\end{align*}
\]

We assume \(BB' = CC'\) and \(C > B\), and lead to contradiction. Since the sum of the interior angles in a hyperbolic triangle is less than \(\pi\), we have \(B < C < \frac{\pi}{2}\), and so, \(\sin B < \sin C\). In the following, we show that \(U < V\) and \(u < v\).

By (2) and (4), we have

\[
\frac{\sinh V}{\sin \gamma} \cdot \sin 2\beta = \frac{\sinh U}{\sin \beta} \cdot \sin 2\gamma,
\]

\[
\frac{\sinh U}{\sinh V} = \frac{\sin \beta \sin 2\beta}{\sin \gamma \sin 2\gamma}.
\]

By (1), we get \(\frac{\sin 2\beta}{\sin 2\gamma} = \frac{\sinh b}{\sinh c}\), so we have the following.

\[
\frac{\sinh U}{\sinh V} = \frac{\sin \beta \sinh b}{\sinh \gamma \sinh c}.
\]

Because of \(\frac{\sin \beta}{\sin \gamma} < 1\) and \(\frac{\sinh b}{\sinh c} < 1\), we get \(\frac{\sin \beta \sinh b}{\sinh \gamma \sinh c} < 1\). Then we have \(\sinh U < \sinh V\). Since the hyperbolic sine function is monotonically increasing, we conclude \(U < V\).
Similarly, by (3) and (5), we have
\[
\frac{\sinh v}{\sin \gamma} = \frac{\sinh u}{\sin \beta}, \\
\frac{\sinh u}{\sinh v} = \frac{\sin \beta}{\sin \gamma} < 1.
\]
Therefore we get \( \sinh u < \sinh v \), that is, \( u < v \).

Now let us consider the ratio and difference of \( \frac{\sinh b}{\sinh u} \) and \( \frac{\sinh c}{\sinh v} \). First we consider the ratio.
\[
\frac{\sinh b}{\sinh u} \cdot \frac{\sinh c}{\sinh v} = \frac{\sinh b \sinh v}{\sinh u \sinh c} = \sinh b \sinh v.
\]
We have the following by (1), (3) and (5).
\[
\frac{\sinh b \sinh v}{\sinh u \sinh c} = \sin \frac{2\beta}{\sinh \gamma} \sin \frac{\gamma}{\sinh \beta}.
\]
Here, we apply the double-angle formula to \( \sin 2\beta, \sin 2\gamma \) respectively.
\[
\frac{\sin 2\beta \sin \gamma}{\sin 2\gamma \sin \beta} = \frac{2 \sin \beta \cos \beta \sin \gamma}{2 \sin \beta \cos \gamma \sin \beta} = \cos \beta.
\]
By assumption \( \beta < \gamma \), we have \( \cos \beta > \cos \gamma \). So \( \frac{\cos \beta}{\cos \gamma} > 1 \). Therefore we get the following result.
\[
(6) \quad \frac{\sinh b}{\sinh u} > \frac{\sinh c}{\sinh v}.
\]
Next we consider the difference, as follows
\[
\frac{\sinh (U + u)}{\sinh u} - \frac{\sinh (V + v)}{\sinh v} = \frac{\sinh U \cosh u + \cosh U \sinh u}{\sinh u} - \frac{\sinh V \cosh v + \cosh V \sinh v}{\sinh v}.
\]
We apply the sum formula to \( \sinh (U + u) \) and \( \sinh (V + v) \) respectively.
\[
\frac{\sinh U}{\sinh u} \cosh u + \cosh U = \sinh V \cosh v + \cosh V \sinh v.
\]
By (4), (5) and (2), (3), \( \frac{\sinh U}{\sinh u} = \frac{\sin A}{\sin 2\gamma} \) and \( \frac{\sinh V}{\sinh v} = \frac{\sin A}{\sin 2\beta} \) hold, and so, we have the following.
\[
\frac{\sinh U}{\sinh u} \cosh u + \cosh U = \sinh V \cosh v + \cosh V \sinh v.
\]
Moreover we get the following by (1).
\[
\frac{\sin A}{\sin 2\gamma} \cosh u + \cosh U = \frac{\sin A}{\sin 2\beta} \cosh v - \cosh V.
\]
By \( \sinh c > \sinh b \), we have \( \frac{\sinh a}{\sinh b} > \frac{\sinh a}{\sinh c} \). And \( \cosh v > \cosh u \) and \( \cosh V > \cosh U \) by \( u < v \) and \( U < V \). Therefore we get the following.
\[
\frac{\sinh a}{\sinh c} \cosh u + \cosh U < \frac{\sinh a}{\sinh b} \cosh v - \cosh V < 0.
\]
Eventually we conclude the following result.
\[
(7) \quad \frac{\sinh b}{\sinh u} < \frac{\sinh c}{\sinh v}.
\]
A contradiction is led by (6) and (7). □
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I would like to referee for comments on an error in the earlier version.

REFERENCES