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A Construction Method of Solutions of Parabolic Systems with Local Hölder Continuity

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Abstract

The aim of this paper is to construct a locally Hölder continuous solution of parabolic systems with spatially continuous coefficients. By means of the discrete Morse flow method, we formulate variational functionals to make approximate solutions with some compactness properties.

1 Introduction

Let T be a positive number and Ω a bounded domain in the Euclidean space $\mathbb{R}^m$ ($m = 2, 3, 4, \dots$) with its boundary $\partial\Omega$ not necessarily smooth. The aim of this paper is to establish a construction method of a locally Hölder continuous solution $u = (u^i)\colon Q \to \mathbb{R}^M$ ($M = 1, 2, 3, \dots$) of the initial-boundary value problem

$$\begin{aligned} \partial_t u^i - D_\alpha(A^{ij}_{\alpha\beta}(x) D_\beta u^j) &= D_\alpha F^i_\alpha \quad \text{in } Q := (0, T) \times \Omega, \\ u &= u_0 \quad \text{on } [\,0, T\,] \times \partial\Omega, \\ u(0, \cdot) &= u_0 \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where $u_0\colon \Omega \to \mathbb{R}^M$ and $F\colon \Omega \to \mathbb{R}^{mM}$ are given. The initial and the lateral boundary conditions are stated in the sense of

$$\lim_{t\downarrow 0} u(t, \cdot) = u_0 \ \text{ in } L^2(\Omega), \quad u(t, \cdot) - u_0 \in H^1_0(\Omega) \ \text{ for a.a. } 0 \le t \le T$$

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respectively. The coefficients $\{A^{ij}_{\alpha\beta}\}$ are assumed to be continuous in $\overline{\Omega}$, the closure of Ω, to be symmetric in the sense of $A^{ij}_{\alpha\beta} = A^{ji}_{\beta\alpha}$ in Ω and to satisfy the Legendre-Hadamard condition with a positive number λ:

$$\lambda|\xi|^2|\eta|^2 \le A^{ij}_{\alpha\beta}(x)\xi_\alpha\xi_\beta\eta^i\eta^j \quad \text{for } x \in \overline{\Omega},\ \xi \in \mathbb{R}^m,\ \eta \in \mathbb{R}^M. \tag{1.2}$$

We adopt Einstein's summation convention for repeated indices: Greek one running from 1 to m and the italic one from 1 to M. It is easier to annihilate the boundary data not only because our arguments are simpler but because Rellich's compactness theorem is applicable without the smoothness of the boundary. For this purpose, we set $v(t,x) := u(t,x) - u_0(x)$ and abbreviate the indices to reduce problem (1.1) to

$$\begin{aligned} \partial_t v - D(A(x)Dv) = D(A(x)Du_0 + F) =: D\widetilde{F} \quad &\text{in } Q,\\ v = 0 \quad &\text{on } [\,0,T\,] \times \partial\Omega,\\ v(0,\cdot) = 0 \quad &\text{in } \Omega. \end{aligned} \tag{1.3}$$

One of the efficient ways to solve parabolic partial differential equations is to construct an approximate solution and pass it to the limit. We discretize the equation in the time variable and consider a sequence of functions $\{v_n\}_{n=0}^N$ inductively satisfying

$$\begin{aligned} v_0 = 0 \quad &\text{in } \Omega,\\ \frac{v_n - v_{n-1}}{h} - D(A(x)Dv_n) = D\widetilde{F} \quad &\text{in } \Omega,\\ v_n = 0 \quad &\text{on } \partial\Omega \quad (n = 1,2,\dots,N). \end{aligned}$$

However, it is still unclear whether the sequence $\{v_n\}$ does exist because, in Gårding's inequality (see [1], [6])

$$\int_\Omega A(D\psi, D\psi)\,dx \ge \frac{\lambda}{4}\int_\Omega |D\psi|^2\,dx - \mu\int_\Omega |\psi|^2\,dx \quad \text{for } \psi \in H^1_0(\Omega), \tag{1.4}$$

induced from the Legendre-Hadamard condition, the term including a positive constant μ has a bad influence on our discussions. We further set $w(t,x) := e^{-\mu t}v(t,x)$ to overcome this difficulty and consider constructing a solution w of the problem

$$\begin{aligned} \partial_t w - D(A(x)Dw) + \mu w = e^{-\mu t}D\widetilde{F} \quad &\text{in } Q,\\ w = 0 \quad &\text{on } [\,0,T\,] \times \partial\Omega,\\ w(0,\cdot) = 0 \quad &\text{in } \Omega. \end{aligned} \tag{1.5}$$

For our convenience, we refer to the term including the factor μ as μ-term. Once a solution w of (1.5) is constructed, we can state our main result of this paper.

Theorem 1 (**Main result**). *Suppose that, for a number γ_0 given between 0 and 1, Du_0 and F belong to the local Morrey space $L^{2,m-2+2\gamma_0}_{\mathrm{loc}}(\Omega)$. Then there exists a locally γ_0-Hölder continuous solution u of (1.1) such that*

$$|u(t,x) - u(t',x')| \le C(|t-t'|^{\gamma_0/2} + |x-x'|^{\gamma_0})$$

for points (t,x) and (t',x') in an arbitrary subdomain $\widetilde{Q} \subset\subset Q$. The constant C depends on the distance between $\partial\Omega$ and $\mathrm{Proj}_{\mathbb{R}^m}\widetilde{Q}$, the orthogonal projection of $\widetilde{Q}$ to $\{t=0\}\times\Omega$.

In Section 2, we construct an approximate solution of (1.5) via the discrete Morse flow method ([11]) for more information on its compactness properties. Section 3 is devoted to the derivation of Campanato decay estimates in local parabolic cylinders $P_r(z_0)$ or $Q_r(z_0)$ for the approximate solution obtained in the previous section. The paper [7] treats only the case where $P_r(z_0)$ is located inside Q. In order to obtain the Hölder continuity of the solution, we add the case where $P_r(z_0)$ crosses the bottom of Q as well, while it is unnecessary that $P_r(z_0)$ touches the lateral boundary. The isomorphism between a Campanato space and a Hölder space is verified in spatial domains due to [2] and in time-space domains with different metric due to [3]. The classical arguments by use of these properties are seen in [5] for elliptic systems and in [13] for parabolic ones. On the other hand, Section 4 proves the local Hölder continuity of the Cauchy-Euler polygon generated from the approximate solution of (1.5) via a *discrete* Morrey space. Morrey estimates are achieved on the basis of [10]; and the local Hölder continuity, on the basis of [12] with some modifications on account of the term μw. In Section 5, we pass approximate solutions to the limit to catch a solution of (1.5) and hence a solution of (1.1). The Hölder continuity corresponding to the Cauchy-Euler polygon is inherited immediately by Ascoli-Arzelà's theorem.

We end this section by introducing the notation used throughout this paper. In the variable $z = (t, x)$, we denote by t a time variable in $[\,0, T\,]$ and by $x = (x_1, x_2, \ldots, x_m)$ a point of $\mathbb{R}^m$. The inner product is symbolized by $x \cdot y$ for $x, y \in \mathbb{R}^d$ $(d \in \mathbb{N})$ and the Euclidean norm of x by $|x| = \{\sum_{i=1}^{d} (x_i)^2\}^{1/2}$. For a positive integer N, we define $h := T/N$ and set

$$t_n = nh \quad (n = 0, 1, 2, \ldots, N), \quad Q_{(h)} = \bigcup_{n=1}^{N} \{t = t_n\} \times \varOmega.$$

For $n_0 = 1, 2, \ldots, N$ and $x_0 \in \varOmega$, we put $z_0 := (t_{n_0}, x_0)$ and write an m-dimensional ball and an $(m+1)$-dimensional parabolic cylinder as

$$B_r(x_0) := \{x \in \mathbb{R}^m : |x - x_0| < r\}, \quad P_r(z_0) := (t_{n_0} - r^2, t_{n_0}) \times B_r(x_0)$$

respectively. In particular, the intersection of $P_r(z_0)$ and Q is symbolized as $Q_r(z_0)$.

For $X, Y \in \mathbb{R}^{mM}$ and coefficients $\{A^{ij}_{\alpha\beta}(x)\}$ $(x \in \varOmega)$, we use the abbreviation

$$X : Y := X^i_\alpha Y^i_\alpha, \quad A(x)(X, Y) := \sum_{\alpha,\beta=1}^{m} \sum_{i,j=1}^{M} A^{ij}_{\alpha\beta}(x) X^i_\alpha Y^j_\beta$$

and define the norm of $A(x)$ as

$$|A| := \max_{x \in \varOmega} \Big(\sum_{\alpha,\beta=1}^{m} \sum_{i,j=1}^{M} |A^{ij}_{\alpha\beta}(x)| \Big)^{1/2}.$$

We shall introduce some function spaces. Denote by $C^\infty_c(\varOmega, \mathbb{R}^M)$ the space of infinitely differentiable functions $u \colon \varOmega \to \mathbb{R}^M$ with compact support in $\varOmega$. The Lebesgue spaces $L^2(\varOmega, \mathbb{R}^M)$ and $L^2(Q, \mathbb{R}^M)$ and the Sobolev space $H^k(\varOmega, \mathbb{R}^M)$ $(k \in \mathbb{N})$ are defined to be the sets of functions u equipped with the finite norms

$$||u||_{L^2(\Omega)} := \Big(\int_\Omega |u|^2\,dx\Big)^{1/2}, \quad ||u||_{L^2(Q)} := \Big(\int_Q |u|^2\,dz\Big)^{1/2},$$
$$||u||_{H^k(\Omega)} := \Big(\int_\Omega |u|^2\,dx + \int_\Omega |Du|^2\,dx + \cdots + \int_\Omega |D^k u|^2\,dx\Big)^{1/2}$$

respectively. The space $H_0^1(\Omega,\mathbb{R}^M)$ is the closure of $C_c^\infty(\Omega,\mathbb{R}^M)$ with respect to the $H^1(\Omega)$-norm. Note that the target space $\mathbb{R}^M$ is omitted if inferred from the context.

We say that a function $f \in L^2(\Omega)$ belongs to the local Morrey space $L^{2,\kappa}_{\mathrm{loc}}(\Omega)$ with a positive number κ if f has a finite norm

$$||f||_{L^{2,\kappa}(\widetilde{\Omega})} := \sup_{\substack{x_0\in\widetilde{\Omega}\\ 0<r<\sigma}} r^{-\kappa}\int_{B_r(x_0)} |f|^2\,dx \tag{1.6}$$

for an arbitrary number $0<\sigma<\mathrm{dist}(\widetilde{\Omega},\partial\Omega)$ and an arbitrary subdomain $\widetilde{\Omega} \subset\subset \Omega$.

2 A related variational problem

For the purpose of the construction of a solution $w \in L^2(Q)$ with $Dw \in L^2(Q)$ of (1.5) in the weak form

$$\int_Q \Big(\partial_t w\cdot\varphi + A(x)(Dw, D\varphi) + \mu w\cdot\varphi\Big)\,dz = -\int_Q e^{-\mu t}\widetilde{F} : D\varphi\,dz \tag{2.1}$$

for $\varphi \in C_c^\infty(Q)$, we construct an approximate solution of (2.1) via the discrete Morse flow method. Let $\{w_n\}_{n=0}^N$ be a sequence of functions in $H_0^1(\Omega)$ as a solution of

$$w_0 = 0 \ \text{ in } \Omega,$$
$$\int_\Omega \Big(\frac{w_n - w_{n-1}}{h}\cdot\varphi + A(x)(Dw_n, D\varphi) + \mu w_n\cdot\varphi\Big)\,dx = -\int_\Omega e^{-\mu t_n}\widetilde{F} : D\varphi\,dx \tag{2.2}$$

for $\varphi \in C_c^\infty(\Omega)$ and $n = 1, 2, \ldots, N$.

In this section, we shall construct a sequence of functions $\{w_n\}_{n=0}^N \subset H_0^1(\Omega)$ satisfying (2.2). For $n = 1, 2, \ldots, N$, we formulate the variational functional

$$I_n(w) := \int_\Omega \Big(\frac{|w - w_{n-1}|^2}{h} + A(x)(Dw, Dw) + \mu|w|^2 + 2e^{-\mu t_n}\widetilde{F} : Dw\Big)\,dx \tag{2.3}$$

in $H_0^1(\Omega)$.

We shall show that (2.3) has a minimizer, which is symbolized as w_n and satisfies (2.2). Since $e^{-\mu t_n} \le 1$ for $n = 1, 2, \ldots, N$ and Gårding's inequality (1.4) cancels the μ-term for $w \in H_0^1(\Omega)$, we have

$$I_n(w) \ge \int_\Omega \frac{|w - w_{n-1}|^2}{h}\,dx + \frac{\lambda}{4}\int_\Omega |Dw|^2\,dx - 2\int_\Omega |\widetilde{F}||Dw|\,dx.$$

Then Young's inequality yields

$$I_n(w) \ge \int_\Omega \frac{|w - w_{n-1}|^2}{h}\,dx + \frac{\lambda}{8}\int_\Omega |Dw|^2\,dx - \frac{8}{\lambda}\int_\Omega |\widetilde{F}|^2\,dx. \tag{2.4}$$

On neglecting nonnegative terms, we obtain the downward boundedness of $I_n(w)$:

$$\inf_{w\in H^1_0(\Omega)} I_n(w) \geq -\frac{8}{\lambda}\int_\Omega |\widetilde{F}|^2\,dx > -\infty.$$

Let $\{w^{(j)}\}_{j=1}^\infty$ be a minimizing sequence of $I_n(w)$. Since, for $j\in\mathbb{N}$, the functional $I_n(w^{(j)})$ is upward bounded as well, we again rely on (2.4) to have

$$\int_\Omega |Dw^{(j)}|^2\,dx \leq \frac{8}{\lambda}\Big(I_n(w^{(j)}) + \frac{8}{\lambda}\int_\Omega |\widetilde{F}|^2\,dx\Big) \leq \frac{8}{\lambda}\Big(\sup_{j\in\mathbb{N}} I_n(w^{(j)}) + \frac{8}{\lambda}\int_\Omega |\widetilde{F}|^2\,dx\Big) < +\infty$$

for all $j\in\mathbb{N}$. It follows from Poincaré's inequality that $\{w^{(j)}\}_{j=1}^\infty$ is bounded in $H^1_0(\Omega)$. Hence Rellich's theorem and the weak compactness of $L^2(\Omega)$ yield a subsequence of $\{w^{(j)}\}$, for which the same notation is used, converging to $w\in H^1_0(\Omega)$ such that

$$w^{(j)}\to w \ \text{ strongly in } L^2(\Omega), \quad Dw^{(j)} \rightharpoonup Dw \ \text{ weakly in } L^2(\Omega) \tag{2.5}$$

as j tends to infinity (see [4]). Note that no smoothness conditions are required on the boundary $\partial\Omega$ because $w^{(j)}$ belongs to $H^1_0(\Omega)$, not just in $H^1(\Omega)$.

Next we claim that w in (2.5) is the minimizer of $I_n(w)$. Before passing $I_n(w^{(j)})$ to the limit $j\to\infty$, we shall demonstrate a lower semi-continuity

$$\varliminf_{j\to\infty}\int_\Omega A(x)(Dw^{(j)},Dw^{(j)})\,dx \geq \int_\Omega A(x)(Dw,Dw)\,dx. \tag{2.6}$$

After a slight transformation by use of the symmetry of $A(x)$, we use Gårding's inequality to have

$$\begin{aligned}
&\int_\Omega A(x)(Dw^{(j)},Dw^{(j)})\,dx - \int_\Omega A(x)(Dw,Dw)\,dx\\
&= \int_\Omega A(x)(Dw^{(j)}-Dw, Dw^{(j)}-Dw)\,dx + 2\int_\Omega A(x)(Dw^{(j)}-Dw,Dw)\,dx\\
&\geq \frac{\lambda}{4}\int_\Omega |Dw^{(j)}-Dw|^2\,dx - \mu\int_\Omega |w^{(j)}-w|^2\,dx + 2\int_\Omega A(x)(Dw^{(j)}-Dw,Dw)\,dx\\
&\geq -\mu\int_\Omega |w^{(j)}-w|^2\,dx + 2\int_\Omega A(x)(Dw^{(j)}-Dw,Dw)\,dx.
\end{aligned}$$

Hence it follows from (2.5) that

$$\varliminf_{j\to\infty}\int_\Omega A(x)(Dw^{(j)},Dw^{(j)})\,dx - \int_\Omega A(x)(Dw,Dw)\,dx \geq 0.$$

This describes (2.6), and hence

$$I_n(w) \leq \varliminf_{j\to\infty} I_n(w^{(j)}) = \inf_{w\in H^1_0(\Omega)} I_n(w).$$

Thus w is the desired minimizer.

We generate an h-step function $w_{(h)}$ from the sequence $\{w_n\}_{n=0}^N$ by

$$w_{(h)}(t) = \begin{cases} w_0 & \text{for } t=0,\\ w_n & \text{for } t_{n-1}<t\leq t_n \quad (n=1,2,\ldots,N). \end{cases} \tag{2.7}$$

We often use the notation $\partial_t w_n$ to stand for the difference quotient

$$\partial_t w_n = \frac{w_n - w_{n-1}}{h} \quad (n = 1, 2, \ldots, N).$$

Lemma 1 below refers to the uniform boundedness of $w_{(h)}$.

Lemma 1 (**uniform boundedness**). *Let* $w_0 = 0$ *in* Ω *and* w_n *be a minimizer of* (2.3) *for* $n = 1, 2, \ldots, N$. *Then* $w_{(h)}$, $Dw_{(h)}$ *and* $\partial_t w_{(h)}$ *are all bounded in* $L^2(Q)$ *uniformly with respect to* $h > 0$:

$$K := \sup_{h>0}\Big(||w_{(h)}||_{L^2(Q)} + ||Dw_{(h)}||_{L^2(Q)} + ||\partial_t w_{(h)}||_{L^2(Q)} \Big) < +\infty. \tag{2.8}$$

Proof. For $n = 0, 1, 2, \ldots, N$ and $w \in H^1_0(\Omega)$, define a functional $J_n(w)$ by

$$J_n(w) := \int_\Omega \Big(A(x)(Dw, Dw) + \mu |w|^2 + 2e^{-\mu t_n} \widetilde{F} : Dw \Big)\, dx.$$

Since w_n minimizes $I_n(w)$, we know $I_n(w_n) \le I_n(w_{n-1})$, that is,

$$\begin{aligned} h||\partial_t w_n||^2_{L^2(\Omega)} + J_n(w_n) &\le J_n(w_{n-1}) \\ &= J_{n-1}(w_{n-1}) + 2(e^{-\mu t_n} - e^{-\mu t_{n-1}}) \int_\Omega \widetilde{F} : Dw_{n-1}\, dx \end{aligned}$$

for $n = 1, 2, \ldots, N$. By iteration and $w_0 = 0$, we obtain

$$\begin{aligned} \sum_{j=1}^{n} h||\partial_t w_j||^2_{L^2(\Omega)} + J_n(w_n) &\le J_0(w_0) + 2\sum_{j=1}^{n} (e^{-\mu t_j} - e^{-\mu t_{j-1}}) \int_\Omega \widetilde{F} : Dw_{j-1}\, dx \\ &= 2\sum_{j=0}^{n-1} (e^{-\mu t_{j+1}} - e^{-\mu t_j}) \int_\Omega \widetilde{F} : Dw_j\, dx \end{aligned} \tag{2.9}$$

for $n = 1, 2, \ldots, N$.

We obviously have $e^{-\mu t_n} \le 1$. On the other hand, by the mean value theorem, we observe

$$|e^{-\mu t_{j+1}} - e^{-\mu t_j}| \le \mu e^{-\mu t_j}(t_{j+1} - t_j) \le \mu h$$

for $j = 1, 2, \ldots, N$. We further use Gårding's and Young's inequalities in (2.9) to obtain

$$\begin{aligned} \sum_{j=1}^{n} h||\partial_t w_j||^2_{L^2(\Omega)} + \frac{\lambda}{8}||Dw_n||^2_{L^2(\Omega)} &\le \frac{8}{\lambda}||\widetilde{F}||^2_{L^2(\Omega)} + 2\sum_{j=0}^{n-1} \mu h \int_\Omega |\widetilde{F}||Dw_j|\, dx \\ &\le \mu \sum_{j=0}^{n-1} h||Dw_j||^2_{L^2(\Omega)} + C||\widetilde{F}||^2_{L^2(\Omega)}. \end{aligned} \tag{2.10}$$

Dropping off the first term of the left-hand side in (2.10), we may regard it as a discrete Gronwall inequality for the nonnegative sequence $\{||Dw_n||^2_{L^2(\Omega)}\}_{n=0}^{N}$ because (2.10) leads to

$$\int_Q |Dw_{(h)}|^2\, dz \le C||\widetilde{F}||^2_{L^2(\Omega)}. \tag{2.11}$$

Combining (2.10) and (2.11), we easily see

$$\int_Q |\partial_t w_{(h)}|^2 \, dz \le \mu \int_Q |Dw_{(h)}|^2 \, dz \le C||\widetilde{F}||^2_{L^2(\Omega)}.$$

Due to $w_{(h)}(t,\cdot) \in H^1_0(\Omega)$, the $L^2(Q)$-boundedness of $w_{(h)}$ follows from Poincaré's inequality in $\mathbb{R}^m$. Thus we can assert (2.8). □

Remark 1. Here is a derivation of (2.11) from (2.10). Putting $\mu' = 8\mu/\lambda$ and $a_n = \sum_{j=0}^n ||Dw_j||^2_{L^2(\Omega)}$, we transform (2.10) into

$$e^{-\mu' t_n} a_n - e^{-\mu' t_{n-1}} a_{n-1} \le (e^{-\mu' t_n} - e^{-\mu' t_{n-1}}) a_{n-1} + \mu' h e^{-\mu' t_n} a_{n-1} + C e^{-\mu' t_n} ||\widetilde{F}||^2_{L^2(\Omega)}.$$

Since the inequality $e^{-\mu' t_n} - e^{-\mu' t_{n-1}} \le -\mu' h e^{-\mu' t_n}$ is obtained from the mean value theorem, the summation with respect to n from 1 to N leads to

$$e^{-\mu' T} a_N \le C \sum_{n=1}^N e^{-\mu' t_n} ||\widetilde{F}||^2_{L^2(\Omega)} \le C \sum_{n=1}^N ||\widetilde{F}||^2_{L^2(\Omega)}$$

where $a_0 = 0$ is due to $Dw_0 = 0$. Hence (2.11) follows from the computation below:

$$||Dw_{(h)}||^2_{L^2(Q)} = h a_N \le C e^{\mu' T} \sum_{n=1}^N h ||\widetilde{F}||^2_{L^2(\Omega)} = C T e^{\mu' T} ||\widetilde{F}||^2_{L^2(\Omega)}.$$

3 Campanato decay estimates for $w_{(h)}$

In this section, we establish Campanato decay estimates for solutions of (2.2) along the framework in [7] and [9].

Lemma 2 (Discrete Campanato decay estimates). *Let $\{w_n\}_{n=0}^N$ be a sequence of solutions of* (2.2). *Suppose that, for a point $x_0 \in \Omega$ and a positive number r, a ball $B_r(x_0)$ is located inside Ω. Then there exists a positive constant $C_0 = C_0(A, \Omega)$ independent of h such that, for $z_0 = (t_{n_0}, x_0) \in Q_{(h)}$ and $0 < \rho \le r$,*

$$\int_{Q_\rho(z_0)} |Dw_{(h)}|^2 \, dz \le C_0 (\rho/r)^{m+2} \int_{Q_r(z_0)} |Dw_{(h)}|^2 \, dz + C_0 \int_{Q_r(z_0)} |A(x) - A(x_0)|^2 |Dw_{(h)}|^2 \, dz + C_0 r^2 \int_{B_r(x_0)} |\widetilde{F}|^2 \, dx, \tag{3.1}$$

provided $r^2 \ge 12([m/2]+4)^2 h$, and

$$\int_{Q_\rho(z_0)} |Dw_{(h)}|^2 \, dz \le C_0 (\rho/r)^{m+2} \int_{Q_r(z_0)} |Dw_{(h)}|^2 \, dz + C_0 \theta (\rho/r)^2 \int_{Q_r(z_0)} |D\widetilde{w}_{(h)}|^2 \, dz + C_0 \int_{Q_r(z_0)} |A(x) - A(x_0)|^2 |Dw_{(h)}|^2 \, dz + C_0 r^2 \int_{B_r(x_0)} |\widetilde{F}|^2 \, dx, \tag{3.2}$$

provided $r^2 \le \theta h$ $(0 < \theta < 12([m/2]+4)^2)$.

Proof. The core observation in the Campanato decay estimates is to estimate solutions of a *homogeneous* system in a half-sized cylinder $P_{r/2}(z_0)$ or a half-sized ball $B_{r/2}(x_0)$. In this situation, Sobolev's imbedding theorem and Caccioppoli inequalities are required.

We write (2.2) as

$$\begin{aligned}\frac{w_n - w_{n-1}}{h} - D(A(x_0)Dw_n) + \mu w_n &= D\big((A(x) - A(x_0))Dw_n + e^{-\mu t_n}\widetilde{F}\big) \\ &=: DG_n.\end{aligned} \tag{3.3}$$

If the coefficients $\{A^{ij}_{\alpha\beta}\}$ are constant, the Legendre-Hadamard condition (1.2) induces a coerciveness

$$\lambda \int_\Omega |D\psi|^2\,dx \le \int_\Omega A(x_0)(D\psi, D\psi)\,dx \quad \text{for } \psi \in H^1_0(\Omega) \tag{3.4}$$

which behaves better than (1.4). By Lax-Milgram's theorem, there exists a sequence of functions $\{f_n\}_{n=p}^{n_0} \subset H^1_0(B_r(x_0))$ for the problem

$$\begin{gathered} f_p = 0, \\ \frac{f_n - f_{n-1}}{h} - D(A(x_0)Df_n) + \mu f_n = DG_n \quad \text{in } B_r(x_0) \quad (n = p+1, p+2, \dots, n_0), \end{gathered} \tag{3.5}$$

where $p = \max\{n_0 - [r^2/h], 0\}$. Choosing hf_n as a test function, we use Cauchy-Schwarz's inequality and (3.4) to have

$$\begin{aligned}&\int_{B_r(x_0)} (|f_n|^2 - |f_{n-1}|^2)\,dx + \lambda h \int_{B_r(x_0)} |Df_n|^2\,dx + 2\mu h \int_{B_r(x_0)} |f_n|^2\,dx \\ &\le \frac{h}{\lambda} \int_{B_r(x_0)} |G_n|^2\,dx\end{aligned}$$

for $n = p+1, p+2, \dots, n_0$. On taking a summation with respect to n, we drop off some nonnegative terms in the left-hand side to obtain

$$\int_{t_p}^{t_{n_0}} \int_{B_r(x_0)} |Df_{(h)}|^2\,dxdt \le \frac{1}{\lambda^2} \int_{t_p}^{t_{n_0}} \int_{B_r(x_0)} |G_{(h)}|^2\,dxdt,$$

where $G_{(h)}$ is an h-step function generated from $\{G_n\}_{n=p}^{n_0}$.

We pay attention to $f_p = 0$ in case of $p = n_0 - [r^2/h] > 0$ and to $Q_r(z_0) = (0, t_{n_0}) \times B_r(x_0)$ in case of $p = 0$ to arrive at

$$\int_{Q_r(z_0)} |Df_{(h)}|^2\,dz \le \frac{1}{\lambda^2} \int_{Q_r(z_0)} |G_{(h)}|^2\,dz. \tag{3.6}$$

On the other hand, $\psi_n := w_n - f_n$ satisfies

$$\frac{\psi_n - \psi_{n-1}}{h} - D(A(x_0)D\psi_n) + \mu\psi_n = 0 \quad \text{in } B_r(x_0) \quad (n = p+1, p+2, \dots, n_0). \tag{3.7}$$

In order to establish the Campanato decay estimate

$$\int_{Q_\rho(z_0)} |D\psi_{(h)}|^2\,dz \le C(\rho/r)^{m+2} \int_{Q_r(z_0)} |D\psi_{(h)}|^2\,dz, \tag{3.8}$$

we need to rely on the dilatation arguments, which allows us to assume $r = 1$ and $x_0 = 0$.

Since (3.8) is trivial for $r/4 \le \rho \le r$, it suffice to give a proof for $0 < \rho < r/4$. For $n_0 = 1, 2, \ldots, N$ and $0 < \rho < r$, we define a discrete cut-off function by a sequence $\{\tau_n\}$ consisting of

$$\tau_n := \tau_{n,n_0;\rho,r} = \begin{cases} 1 & \text{for } n_0 - [\rho^2/h] \le n \le n_0, \\ \dfrac{n - n_0 + [r^2/h] - 1}{[r^2/h] - [\rho^2/h] - 1} & \text{for } n_0 - [r^2/h] + 2 \le n \le n_0 - [\rho^2/h] - 1, \\ 0 & \text{for } n \le n_0 - [r^2/h] + 1, \end{cases} \tag{3.9}$$

which is well defined under the condition $(r - \rho)^2 \ge 3h$ with the property

$$0 \le \tau_n - \tau_{n-1} \le 3h/(r - \rho)^2 \quad (n = 1, 2, \ldots, n_0). \tag{3.10}$$

According to the regularity theory for elliptic partial differential equations ([4]), ψ_n is seen to have weak derivatives of higher order than necessary in shrunk balls (see [7], [9]). We differentiate (3.7) and multiply it by $2\tau_n h D\psi_n$ ($\tau_n = \tau_{n,n_0;1/4,1/2}$). Cauchy-Schwarz's inequality, (3.9) and (3.10) yield

$$\begin{aligned} &\tau_h|D\psi_n|^2 - \tau_{n-1}|D\psi_{n-1}|^2 + 2\mu\tau_n h|D\psi_n|^2 \\ &\le (\tau_n - \tau_{n-1})|D\psi_{n-1}|^2 + |A|\tau_n h(|D\psi_n|^2 + |D^3\psi_n|^2) \\ &\le 48h|D\psi_{n-1}|^2 + |A|h(|D\psi_n|^2 + |D^3\psi_n|^2). \end{aligned}$$

Here, by dropping off the μ-term, we can trace the proofs in [7] and [9] to get

$$\int_{Q_\rho(z_0)} |D\psi_{(h)}|^2\, dz \le C(\rho/r)^{m+2} \sum_{k=1}^{[m/2]+4} \int_{Q_{1/2}(z_0)} |D^k\psi_{(h)}|^2\, dz$$

for $0 < \rho < 1/4$. Hence (3.8) follows from the Caccioppoli inequality

$$\int_{Q_{1/2}(z_0)} |D^k\psi_{(h)}|^2\, dz \le C \int_{Q_1(z_0)} |D\psi_{(h)}|^2\, dz \quad (k \ge 2) \tag{3.11}$$

and scaling back.

We shall show (3.11) independently of μ. Let $\chi \in C_c^\infty(B_1(0))$ be a cut-off function with properties

$$\chi \equiv 1 \ \text{ in } B_1(0), \quad 0 \le \chi \le 1, \quad |D\chi| \le 4. \tag{3.12}$$

We adopt $2h\tau_n\chi^2\psi_n$ ($\tau_n = \tau_{n,n_0;1/2,1}$) as a test function in (3.7). Since the coerciveness (3.4) is valid for $\chi\psi_n$ in $B_1(0)$ (refer to [5]), we use the algebraic equality

$$\begin{aligned} A(0)(D(\chi\psi_n), D(\chi\psi_n)) &= \chi^2 A(0)(D\psi_n, D\psi_n) \\ &\quad + 2\chi A(0)(D\psi_n, D\chi \otimes \psi_n) + A(0)(D\chi \otimes \psi_n, D\chi \otimes \psi_n), \end{aligned} \tag{3.13}$$

and Cauchy-Schwarz's inequality to have

$$\int_{B_1(0)} (\tau_n|\psi_n|^2 - \tau_{n-1}|\psi_{n-1}|^2)\,dx + 2\tau_n h \int_{B_1(0)} A(0)(D(\chi\psi_n), D(\chi\psi_n))\,dx$$
$$+ \mu\tau_n h \int_{B_1(0)} \chi^2|\psi_n|^2\,dx$$
$$\leq (\tau_n - \tau_{n-1}) \int_{B_1(0)} \chi^2|\psi_{n-1}|^2\,dx + |A|\tau_n h \int_{B_1(0)} |D\chi|^2|\psi_n|^2\,dx$$

Here we drop off the μ-term and take a summation with respect to n from $n_0 - [1/h] + 2$ or 1 to n_0. Taking into account (3.9), (3.10) and (3.12), we trace the proofs in [7] and [9] to obtain

$$\int_{Q_{1/2}(z_0)} |D\psi_{(h)}|^2\,dz \leq C \int_{Q_1(z_0)} |\psi_{(h)}|^2\,dz.$$

The homogeneity of (3.7) indicates that the sequence $\{D^l\psi_n\}$ satisfies (3.7) in the ball of radius $r_l = 1 - l/2k$ $(l = 0, 1, 2, \ldots, k)$. Hence, with a slight modification, we have

$$\int_{Q_{r_{l+1}}(z_0)} |D^{l+1}\psi_{(h)}|^2\,dz \leq Ck^2 \int_{Q_{r_l}(z_0)} |D^l\psi_{(h)}|^2\,dz$$

for $l = 1, 2, \ldots, k-1$. Thus (3.11) follows by iteration from $l = k-1$ down to $l = 1$ and inflation of the integral range $Q_{r_1}(z_0)$ to $Q_1(z_0)$.

We shall estimate $w_{(h)} = f_{(h)} + \psi_{(h)}$ by way of

$$\int_{Q_\rho(z_0)} |Dw_{(h)}|^2\,dz \leq 2 \int_{Q_\rho(z_0)} (|Df_{(h)}|^2 + |D\psi_{(h)}|^2)\,dz. \tag{3.14}$$

We substitute (3.6) and (3.8) into (3.14) to have

$$\int_{Q_\rho(z_0)} |Dw_{(h)}|^2\,dz \leq C(\rho/r)^{m+2} \int_{Q_r(z_0)} |Dw_{(h)}|^2\,dz + C \int_{Q_r(z_0)} |G_{(h)}|^2\,dz.$$

Recalling $G_n = (A(x) - A(x_0))Dw_n + e^{-\mu t_n}\widetilde{F}$ and $e^{-\mu t_n} \leq 1$, we complete the proof of (3.1).

Next we go on to the proof of (3.2). For each $n = 1, 2, \ldots, N$, we write (2.2) as

$$\frac{w_n}{h} - D(A(x_0)Dw_n) + \mu w_n = \frac{w_{n-1}}{h} + DG_n.$$

Lax-Milgram's theorem produces a solution $f_n \in H^1_0(B_r(x_0))$ of the problem

$$\frac{f_n}{h} - D(A(x_0)Df_n) + \mu f_n = \frac{w_{n-1} - (w_{n-1})_{r,x_0}}{h} + DG_n \quad \text{in } B_r(x_0),$$

where $(w_{n-1})_{r,x_0}$ is the average of w_{n-1} over the ball $B_r(x_0)$. Choosing f_n as a test function, we can drop off $|f_n|^2$-integrated terms after using Cauchy-Schwarz's inequality.

The coerciveness (3.4), Poincaré's inequality and $r^2 \leq \theta h$ yield

$$\int_{B_r(x_0)} |Df_n|^2\,dx \leq C\theta \int_{B_r(x_0)} |Dw_{n-1}|^2\,dx + C \int_{B_r(x_0)} |G_n|^2\,dx \tag{3.15}$$

for $n = 1, 2, \ldots, N$.

On the other hand, $\psi_n := w_n - f_n$ satisfies

$$\frac{\psi_n}{h} - D(A(x_0)D\psi_n) + \mu\psi_n = \frac{(w_{n-1})_{r,x_0}}{h} \quad \text{in } B_r(x_0). \tag{3.16}$$

We shall establish a Campanato decay estimate

$$\int_{B_\rho(x_0)} |D\psi_n|^2 \, dx \le C(\rho/r)^m \int_{B_r(z_0)} |D\psi_n|^2 \, dx. \tag{3.17}$$

Since (3.17) is trivial for $r/4 \le \rho \le r$, it suffice to give a proof for $0 < \rho < r/4$. We need to rely on the dilatation arguments, which allows us to assume $r = 1$ and $x_0 = 0$ without loss of generality.

The function ψ_n here is also seen to have weak derivatives of higher order than necessary in shrunk balls (see [7], [9]). We rely on Sobolev's imbedding theorem to have

$$\int_{B_\rho(0)} |D\psi_n|^2 \, dx \le C(\rho/r)^m \sum_{k=1}^{[m/2]+2} \int_{B_{1/2}(0)} |D^k\psi_n|^2 \, dx$$

for $0 < \rho < 1/4$. Hence (3.17) follows from the Caccioppoli inequality

$$\int_{B_{1/2}(0)} |D^k\psi_n|^2 \, dz \le C \int_{B_1(0)} |D\psi_n|^2 \, dz \quad (k \ge 2) \tag{3.18}$$

and scaling back.

On differentiating both sides of (3.16), we find $D\psi_n$ to be a solution of the *homogeneous* system

$$\frac{D\psi_n}{h} - D(A(0)D(D\psi_n)) + \mu D\psi_n = 0 \tag{3.19}$$

in any shrunk balls. Let $\chi \in C_c^\infty(B_{3/4}(0))$ be a cut-off function having the similar properties to (3.12). Adopting $\chi^2 D\psi_n$ as a test function, we use the algebraic equality (3.13) to have

$$\begin{aligned} &\frac{1}{h}\int_{B_{3/4}(0)} |D\psi_n|^2 \, dx + 2\int_{B_{3/4}(0)} A(0)(D(\chi D\psi_n), D(\chi D\psi_n)) \, dx + \mu \int_{B_{3/4}(0)} \chi^2 |D\psi_n|^2 \, dx \\ &\le |A| \int_{B_{3/4}(0)} |D\chi|^2 |D\psi_n|^2 \, dx. \end{aligned}$$

In the left-hand side, the μ-term can be neglected in particular. Hence we trace the proofs in [7] and [9] to obtain

$$\int_{B_{1/2}(0)} |D^2\psi_n|^2 \, dx \le 16C \int_{B_{3/4}(0)} |D\psi_n|^2 \, dx.$$

Since $D^l\psi_n$ is also a solution of (3.19) in the ball of radius $r_l = 1 - l/2k$ $(l = 1, 2, \ldots, k)$, the same arguments lead to

$$\int_{B_{r_{l+1}(0)}} |D^{l+1}\psi_n|^2 \, dx \le Ck^2 \int_{B_{r_l}(0)} |D^l\psi_n|^2 \, dx.$$

for $l = 1, 2, \ldots, k-1$. On iteration from $l = k-1$ down to $l = 1$, we arrive at (3.18).

We shall estimate $w_{(h)} = f_{(h)} + \psi_{(h)}$ by way of

$$\int_{B_\rho(x_0)} |Dw_n|^2\,dx \le 2\int_{B_\rho(x_0)} (|Df_n|^2 + |D\psi_n|^2)\,dx. \tag{3.20}$$

We substitute (3.15) and (3.17) into (3.20) to have

$$\begin{aligned}&\int_{B_\rho(x_0)} |Dw_n|^2\,dx\\ &\le C(\rho/r)^m \int_{Q_r(z_0)} |Dw_n|^2\,dx + C\theta\int_{B_r(x_0)} |Dw_{n-1}|^2\,dx + C\int_{B_r(x_0)} |G_n|^2\,dx.\end{aligned}$$

Recalling $G_n = (A(x) - A(x_0))Dw_n + e^{-\mu t_n}\widetilde{F}$ and $e^{-\mu t_n} \le 1$, we complete the proof of (3.2) through the same arguments in [7] and [9]. □

4 Local Morrey estimates and Local Hölder continuity

Our eventual goal of this section is to build a local Hölder continuity of $w^*_{(h)} : [\,0, T\,] \to H^1_0(\varOmega)$, the Cauchy-Euler polygon generated from $\{w_n\}_{n=0}^N$. More precisely, both the Hölder exponent and the Hölder estimate are independent of h, and $w^*_{(h)}$ is defined as

$$w^*_{(h)}(t) := \frac{t_{n+1} - t}{h}\,w_n + \frac{t - t_n}{h}\,w_{n+1} \quad \text{for } t \in [\,t_n, t_{n+1}\,] \quad (n = 0, 1, \ldots, N).$$

Our derivation plan is along the lines in [10]. First, we obtain the Morrey estimate of $Dw_{(h)}$. Secondly, we get the Campanato estimate through the Poincaré-type inequality. Finally, we achieve the local Hölder continuity by invoking the discrete Campanato theory shown in [12].

Let $\widetilde{Q}$ be a subdomain of Q and $\widetilde{\varOmega}$ its orthogonal projection to the plane $\{t = 0\} \times \varOmega$. We put $\widetilde{Q}_{(h)} := \widetilde{Q} \cap Q_{(h)}$ and designate by $d_{\widetilde{\varOmega}}$ the distance in $\mathbb{R}^m$ between $\widetilde{\varOmega}$ and $\partial\varOmega$, which is positive since $\widetilde{\varOmega}$ is a subdomain compactly contained in $\varOmega$. By the same method as used in [10], we first establish the *local* Morrey estimate, namely, we estimate the following local norm:

$$||Du_{(h)}||_{2, m+2\gamma_0, \widetilde{Q}_{(h)}, \sigma} = \sup_{\substack{x_0 \in \widetilde{\varOmega} \\ n = 1, 2, \ldots, N \\ 0 < r < \sigma}} r^{-(m+2\gamma_0)} \int_{Q_r(t_n, x_0)} |Du_{(h)}|^2\,dz, \tag{4.1}$$

where σ is an arbitrarily fixed positive constant smaller than $d_{\widetilde{\varOmega}}$. The remark is that our arguments starts from the Campanato decay estimates in Lemma 2, do not trace back from the structure of the equation (2.2). Consequently, we modify two sides of the proof in [10]. The first one is to use the following local value in order to estimate the non-homogeneous term

$$L(\widetilde{\varOmega}, \sigma) := \sup_{\substack{x_0 \in \widetilde{\varOmega} \\ 0 < r < \sigma}} r^{-(m-2+2\gamma_0)} \int_{B_r(x_0)} \Big(|Du_0|^2 + |F|^2\Big)\,dx \quad (0 < \sigma < d_{\widetilde{\varOmega}}),$$

which is finite since Du_0 and F are assumed to belong to $L^{2,m-2+2\gamma_0}_{\mathrm{loc}}(\Omega)$. While the second one is to restrict the radius of considered local cylinders to positive numbers less than $d_{\widetilde{\Omega}}$. As a result, we can observe

Lemma 3 (**Local Morrey estimate for** $Dw_{(h)}$). *For any* $0 < \sigma < d_{\widetilde{\Omega}}$, $Dw_{(h)}$ *has a finite norm defined in* (4.1)*:*

$$||Dw_{(h)}||_{2,m+2\gamma_0,\widetilde{Q}_{(h)},\sigma} \le C_\sigma,$$

where C_σ *is a positive constant determined depending on* m, A, Ω, T, γ_0, σ, $L(\widetilde{\Omega},\sigma)$, $||Du_0||_{L^2(\Omega)}$ *and* $||F||_{L^2(\Omega)}$.

We notice that, for the proof, we have used the global L^2 estimate of $Dw_{(h)}$ on Q which is shown in Lemma 1.

We proceed our arguments to Poincaré's inequality corresponding to that in [12, Lemma 2.3]. In order to derive this, our computation relies on the structure of the equation (2.2), which requires us to have good control of the μ-term the paper [12] does not have. Let us describe Poincaré's inequality and the modifications of its proof. Let $\sigma \in C_c^\infty(B_1(0))$ be a nonnegative and nonzero function such that

$$||\sigma||_\infty := \max_{x\in B_1(0)} |\sigma(x)| < +\infty, \quad |D\sigma| \le 2 \text{ in } B_1(0).$$

We write

$$\gamma := \int_{B_1(0)} \sigma\, dx, \qquad\qquad \gamma' := \int_{B_1(0)} \sigma^2\, dx,$$

which are positive. Now for $x_0 \in \mathbb{R}^m$ and $r > 0$, we define

$$\sigma_r(x) := \sigma\Big(\frac{x - x_0}{r}\Big) \text{ for } x \in \mathbb{R}^m.$$

Lemma 4 (**Estimate for integral avarage**). *Let* $\{w_n\}_{n=1}^N$ *be a solution of* (2.2)*. Then there exists a positive constant* $C = C(|A|, \mu)$ *such that*

$$|w^{\sigma_r}_{n,r} - w^{\sigma_r}_{n',r}|^2 \le Cr^{-m} \int_{P_r(z_0)} \Big(|Dw_{(h)}|^2 + |\widetilde{F}(x)|^2\Big)\, dz + Cr^{2-m} \int_{P_r(z_0)} |w_{(h)}|^2 dz$$

for $z_0 = (t_{n_0}, x_0) \in Q_{(h)}$, $r > 0$ *and* $n_0 - [r^2/h] \le n' < n \le n_0$ *with* $P_r(z_0) \subset Q$, *where* $w^{\sigma_r}_{n,r}$ *is the weighted integral average of* w_n*:*

$$w^{\sigma_r}_{n,r} := \Big(\int_{B_r(x_0)} \sigma_r\, dx\Big)^{-1} \int_{B_r(x_0)} w_n \sigma_r\, dx.$$

Proof. By substituting $h(w^{\sigma_r}_{n,r} - w^{\sigma_r}_{n',r})\sigma_r(x)$ as a test function of the equation (2.2) at $n = k$, we have

$$\begin{aligned}&(w^{\sigma_r}_{n,r} - w^{\sigma_r}_{n',r}) \int_\Omega (w_k - w_{k-1})\sigma_r(x)\, dx \\ &= -h \int_\Omega (w^{\sigma_r}_{n,r} - w^{\sigma_r}_{n',r})(A(x)Dw_k + e^{-\mu t_k}\widetilde{F}(x))D\sigma_r\, dx - h\mu \int_\Omega w_k\sigma_r(w^{\sigma_r}_{n,r} - w^{\sigma_r}_{n',r})\, dx\end{aligned}$$

for $k = 1, 2, \dots, N$ Taking a summation with respect to k from $n'+1$ to n, we obtain

$$\begin{aligned}
&(w^{\sigma_r}_{n,r} - w^{\sigma_r}_{n',r}) \int_{B_r(x_0)} (w_n - w_{n'})\sigma_r(x)\,dx \\
&= - \sum_{n=n'+1}^{n} h \int_{B_r(x_0)} (A(x)Dw_k + e^{-\mu t_k}\widetilde{F}(x))D\sigma_r(w^{\sigma_r}_{n,r} - w^{\sigma_r}_{n',r})\,dx \\
&\quad - \mu \sum_{n=n'+1}^{n} h \int_{B_r(x_0)} w_n\sigma_r(w^{\sigma_r}_{n,r} - w^{\sigma_r}_{n',r})\,dx
\end{aligned}$$

The definition of $w^{\sigma_r}_{n,r}$ and $w^{\sigma_r}_{n',r}$ reveal

$$\begin{aligned}
\gamma r^m |w^{\sigma_r}_{n,r} - w^{\sigma_r}_{n',r}|^2 \le C(A) &\sum_{n=n'+1}^{n} h \int_{B_r(x_0)} \frac{1}{r}\Big(|Dw_k| + |\widetilde{F}(x)|\Big)|w^{\sigma_r}_{n,r} - w^{\sigma_r}_{n',r}|\,dx \\
&+ C(\mu, ||\sigma||_\infty) \sum_{n=n'+1}^{n} h \int_{B_r(x_0)} |w_k||w^{\sigma_r}_{n,r} - w^{\sigma_r}_{n',r}|\,dx,
\end{aligned}$$

where we estimate $A(x)$ by $|A|$ and use the inequalities $e^{-\mu t_k} \le 1$, $|D\sigma_r| \le 2/r$ and $|\sigma_r| \le ||\sigma||_\infty$. By representing summation by the integrals, we infer

$$\begin{aligned}
&\gamma r^m |w^{\sigma_r}_{n,r} - w^{\sigma_r}_{n',r}|^2 \\
&\le C \int_{P_r(z_0)} \frac{1}{r}\Big(|Dw_{(h)}| + |\widetilde{F}(x)|\Big)|w^{\sigma_r}_{n,r} - w^{\sigma_r}_{n',r}|\,dz + C \int_{P_r(z_0)} |w_{(h)}||w^{\sigma_r}_{n,r} - w^{\sigma_r}_{n',r}|\,dz,
\end{aligned}$$

where $C = C(|A|, ||\sigma||_\infty, \mu)$. We here apply Young's inequality to the right-hand side, so that

$$r^m |w^{\sigma_r}_{n,r} - w^{\sigma_r}_{n',r}|^2 \le C \int_{P_r(z_0)} \Big(|Dw_{(h)}|^2 + |\widetilde{F}(x)|^2\Big)\,dz + Cr^2 \int_{P_r(z_0)} |w_{(h)}|^2\,dz.$$

We arrive at the conclusion by dividing the both sides by r^m. □

By the same way as for the proof of [12, Lemma 2.3], we establish the following Poincaré-type inequality with $|w_{(h)}|^2$-integrated term.

Lemma 5 (**Poincaré-type inequality**). *Let* $\{w_n\}_{n=1}^N$ *be a solution of* (2.2). *Then there exists a positive constant* $C = C(m, |A|) > 0$ *such that*

$$\int_{P_r(z_0)} |w_{(h)}(z) - w_{(h),r}|^2\,dz \le C\Big\{r^2 \int_{P_r(z_0)} (|Dw_{(h)}|^2 + |\widetilde{F}|^2)\,dz + r^4 \int_{P_r(z_0)} |w_{(h)}|^2\,dz\Big\}.$$

for all $h > 0$, $x_0 \in \Omega$, $z_0 = (t_{n_0}, x_0) \in Q_{(h)}$ *and* $r > 0$ *with* $P_r(z_0) \subset Q$.

We here in particular notice that the integral of $|w_{(h)}|^2$ is multiplied by r to the fourth power because of the result of Lemma 4.

By tracing the arguments in [8], Lemma 4 holds up to the cylinders $P_r(z_0)$ crossing the bottom of Q only, and so is Lemma 5.

Once the integral of $|w_{(h)}|^2$ is estimated by r to the $(m + 2\gamma_0)$-th power, we can immediately lead the required Campanato estimate by resorting to Poincaré's inequality above.

Proposition 1 (Local Morrey estimate for $w_{(h)}$). *Let $\widetilde{Q} \subset\subset Q$ be a subdomain and $\widetilde{\Omega} \subset\subset \Omega$ be its orthogonal projection of $\widetilde{Q}$ to $\{t=0\} \times \Omega$. For any $0<\sigma<d_{\widetilde{\Omega}}$, there exists a positive constant C_σ such that*

$$\int_{Q_r(t_n,x_0)} |w_{(h)}|^2 \, dz \le C_\sigma r^{m+2\gamma_0} \quad \text{for } x_0 \in \widetilde{\Omega}, \quad n=1,2,\dots,N, \quad 0<r<\sigma.$$

Proof. The route to accomplish the proof is similar to that of Lemma 2. So we state only the outline. To begin with, we aim at the Campanato type inequality for $w_{(h)}$. Let $x_0 \in \Omega$ and consider the cylinder $P_r(z_0)$, where $0<r<\operatorname{dist}(x_0,\partial\Omega)$, $z_0=(t_{n_0},x_0) \in Q_{(h)}$. We consider (2.2) in the form (3.3).

Case 1: $r^2 \ge 12([m/2]+4)^2 h$.
Let f_n and ψ_n be solutions of (3.5) and (3.7), respectively. As in the proof of Lemma 2. By establishing Campanato decay estimate for ψ_n and a simple estimate for f_n, we add them to have

$$\begin{aligned} \int_{Q_\rho(z_0)} |w_{(h)}|^2 \, dz &\le C(\rho/r)^{m+2} \int_{Q_r(z_0)} |w_{(h)}|^2 \, dz \\ &+ Cr^2 \int_{Q_r(z_0)} |A(x)-A(x_0)|^2 |Dw_{(h)}|^2 \, dz + Cr^4 \int_{B_r(x_0)} (|Du_0|^2+|F|^2) \, dx. \end{aligned} \tag{4.2}$$

Case 2: $r^2 \le \theta h$ $(\theta \le 12([m/2]+4)^2)$.
Differently from the proof of Lemma 2, an integral average is unnecessary in the decomposition of w_n since our goal is to perform the Campanato decay estimate not for Dw_n but for w_n itself. For $n=1,2,\dots,N$, let f_n be a solutions of

$$\begin{cases} f_n = 0 & \text{on } \partial B_r(x_0), \\ \dfrac{f_n}{h} - D(A(x_0)Df_n) + \mu f_n = \dfrac{w_{n-1}}{h} + DG_n & \text{in } B_r(x_0), \end{cases} \tag{4.3}$$

and put $\psi_n := w_n - f_n$, which satisfies

$$\begin{cases} \psi_n = w_n & \text{on } \partial B_r(x_0), \\ \dfrac{\psi_n}{h} - D(A(x_0)D\psi_n) + \mu\psi_n = 0 & \text{in } B_r(x_0). \end{cases} \tag{4.4}$$

Choosing $\chi^2\psi_n$ as a test function in the weak form of the equation (4.4), we have

$$\int_{B_\rho(x_0)} |\psi_n|^2 \, dx \le C(\rho/r)^m \int_{B_r(x_0)} |\psi_n|^2 \, dx.$$

On the other hand, we choose f_n itself as a test function in the weak form of the equation (4.3), so that

$$\begin{aligned} \int_{B_r(x_0)} |f_n|^2 \, dx &\le C\theta \int_{B_r(x_0)} |w_{n-1}|^2 \, dx \\ &\quad + Cr^2 \int_{B_r(x_0)} \Big(|A(x)-A(x_0)|^2 |Df_n|^2 + |Du_0|^2 + |F|^2 \Big) \, dx. \end{aligned}$$

The combination of the last two estimates gives us

$$\int_{B_\rho(x_0)} |w_n|^2\,dx \le C(\rho/r)^m \int_{B_r(x_0)} |w_n|^2\,dx + C\theta \int_{B_r(x_0)} |w_{n-1}|^2\,dx + Cr^2 \int_{B_r(x_0)} \Big(|A(x)-A(x_0)|^2|Dw_n|^2 + |Du_0|^2 + |F|^2\Big)\,dx.$$

By the same arguments as illustrated in Section 3, we obtain

$$\begin{aligned}\int_{Q_\rho(z_0)} |w_{(h)}|^2\,dz &\le C(\rho/r)^{m+2} \int_{Q_r(z_0)} |w_{(h)}|^2\,dz + C\theta(\rho/r)^2 \int_{Q_r(z_0)} |\widetilde{w}_{(h)}|^2\,dz \\ &+ Cr^2 \int_{Q_r(z_0)} |A(x)-A(x_0)|^2|Dw_{(h)}|^2\,dz + Cr^4 \int_{B_r(x_0)} \Big(|Du_0|^2 + |F|^2\Big)\,dx.\end{aligned} \tag{4.5}$$

In (4.2), taking the Morrey estimate for $Dw_{(h)}$ shown in Lemma 3 and the assumptions that Du_0, $F \in L^{2,m-2+2\gamma_0}_{\mathrm{loc}}(\Omega)$ into account, we arrive at

$$\int_{Q_\rho(z_0)} |w_{(h)}|^2\,dz \le C(\rho/r)^{m+2} \int_{Q_r(z_0)} |w_{(h)}|^2\,dz + Cr^{m+2\gamma_0}.$$

Likewise we get from (4.5)

$$\int_{Q_\rho(z_0)} |w_{(h)}|^2\,dz \le C(\rho/r)^{m+2} \int_{Q_r(z_0)} |w_{(h)}|^2\,dz + C\theta(\rho/r)^2 \int_{Q_r(z_0)} |\widetilde{w}_{(h)}|^2\,dz + Cr^{m+2\gamma_0}.$$

These two inequalities correspond to those in Lemma 2, and hence by the same arguments as in Lemma 3 we come to the conclusion. □

By applying Lemma 3 and Proposition 1 together with the Poincaré-type inequality in Lemma 5, we reach a *local Campanato estimate*

$$\int_{P_r(z_0)} |w_{(h)}(z) - w_{(h),r}|^2\,dz \le Cr^{m+2+2\gamma_0},$$

for $z_0 \in Q_{(h)}$ and $0 < r < \sigma$. As a result, employing the discrete Campanato theory established in [12], we achieve the local Hölder continuity of the Cauchy-Euler polygon $w^*_{(h)}$ in the sense that

$$|w^*_{(h)}(t,x) - w^*_{(h)}(t',x')| \le C(|t-t'|^{\gamma_0/2} + |x-x'|^{\gamma_0}) \ \text{ for } (t,x),\ (t',x') \in \widetilde{Q}_{(h)}. \tag{4.6}$$

Finally, a simple computation guarantees us that $\widetilde{Q}_{(h)}$ in (4.6) is replaced by $\widetilde{Q}$, which is the desired conclusion of this section.

5 Proof of the main result

We shall observe the convergence of $w_{(h)}$ as h tends to zero. If necessary, we choose a subsequence $\{w_{(h_j)}\}_{j=1}^\infty$ and use the same symbol.

According to Lemma 1, there exists a subsequence of $\{w_{(h)}\}$ convergent to a function w such that

$$w_{(h)} \to w \quad \text{strongly in } L^2(Q), \tag{5.1}$$
$$Dw_{(h)} \rightharpoonup Dw \quad \text{weakly in } L^2(Q), \tag{5.2}$$
$$\partial_t w_{(h)} \rightharpoonup \partial_t w \quad \text{weakly in } L^2(Q). \tag{5.3}$$

The convergence (5.1) is due to Rellich's compactness theorem which is valid without any smoothness conditions on $\partial\Omega$ since $w_{(h)}(t,\cdot)$ belongs to $H_0^1(\Omega)$. The weak convergences (5.2) and (5.3) are easily seen.

We further define a function $t_{(h)}(t)$ as

$$t_{(h)}(t) = t_n \quad \text{for } t_{n-1} < t \le t_n \quad (n = 0, 1, 2, \ldots, N).$$

Then $t_{(h)}$ converges to t uniformly as h approaches zero. Indeed, it holds that

$$\sup_{0\le t\le T} |t_{(h)}(t) - t| = \sup_{\substack{t_{n-1}<t\le t_n \\ n=1,2,\ldots,N}} |t_n - t| < h \to 0 \quad (h \to 0). \tag{5.4}$$

By use of (2.8) and (5.2)–(5.4), we pass each term of the identity

$$\int_Q \left(\partial_t w_{(h)} \cdot \varphi + A(x)(D\varphi, Dw_{(h)}) + \mu w_{(h)} \cdot \varphi\right) dz = -\int_Q e^{-\mu t_{(h)}} \widetilde{F} : D\varphi \, dz$$

to the limit as $h \to 0$ for $\varphi \in C_c^\infty(Q)$. Since the strong convergence (5.1) implies the weak convergence, each term of the left-hand side results in

$$\int_Q \partial_t w_{(h)} \cdot \varphi \, dz \to \int_Q \partial_t w \cdot \varphi \, dz,$$
$$\int_Q A(x)(D\varphi, Dw_{(h)}) \, dz \to \int_Q A(x)(D\varphi, Dw) \, dz,$$
$$\int_Q \mu w_{(h)} \cdot \varphi \, dz \to \int_Q \mu w \cdot \varphi \, dz.$$

On the other hand, the convergence (2.8) and the mean value theorem imply that $e^{-\mu t_{(h)}(t)}$ converges to $e^{-\mu t}$ uniformly as $h \to 0$. Indeed, as h tends to zero, we have

$$|e^{-\mu t_{(h)}(t)} - e^{-\mu t}| \le \mu e^{-\mu t} |t_{(h)}(t) - t| \le \mu |t_{(h)}(t) - t| < \mu h \to 0 \quad (h \to 0)$$

for all $0 \le t \le T$. Hence it follows that

$$\lim_{h\to 0} \int_Q e^{-\mu t_{(h)}(t)} \widetilde{F} : D\varphi \, dz = \int_Q e^{-\mu t} \widetilde{F} : D\varphi \, dz.$$

Thus w satisfy the equation (2.1) for all $\varphi \in C_c^\infty(Q)$.

We shall check out the initial and the boundary conditions in (1.5). Since (2.8) and (5.3) imply

$$||\partial_t w||_{L^2(Q)} \le \varliminf_{h\to 0} ||\partial_t w_{(h)}||_{L^2(Q)} \le K,$$

Schwarz's inequality yields

$$\begin{aligned} ||w(s)-w(t)||_{L^2(\Omega)} &= \Big(\int_\Omega \Big|\int_t^s \partial_t w(\tau)\,d\tau\Big|^2\,dx\Big)^{1/2} \\ &\le \Big(\int_\Omega (s-t)\int_t^s |\partial_t w(\tau)|^2\,d\tau\,dx\Big)^{1/2} \\ &\le (s-t)^{1/2}||\partial_t w||_{L^2(Q)} \le K(s-t)^{1/2} \end{aligned}$$

for $0<t<s<T$. As a result of Cauchy's test, $||w(t)||_{L^2(Q)}$ converges in some positive number as $t\downarrow 0$.

We make a similar calculation with attention to $w_{(h)}(0)=0$ to find

$$||w_{(h)}(t)||_{L^2(\Omega)} \le t^{1/2}||\partial_t w_{(h)}||_{L^2(Q)} \le Ct^{1/2} \tag{5.5}$$

for $0<t<T$ and $h>0$.

On the other hand, the convergence (5.1) admits a subsequence of $\{w_{(h)}\}_{h>0}$ and a null set $\Sigma\subset(0,T)$ such that

$$\lim_{h\to 0}||w_{(h)}(t)-w(t)||_{L^2(\Omega)}=0 \quad \text{for } t\notin\Sigma. \tag{5.6}$$

On combination of Minkowski's inequality, (5.5) and (5.6), we can assert that, for an arbitrary positive number ε, there exist positive numbers $t_0\notin\Sigma$ and h_0 both close to 0 such that

$$||w(t_0)||_{L^2(\Omega)} \le ||w(t_0)-w_{(h_0)}(t_0)||_{L^2(\Omega)} + ||w_{(h_0)}(t_0)||_{L^2(\Omega)} < \varepsilon.$$

For $\varepsilon=1/j$ $(j\in\mathbb{N})$ in particular, we choose a number $t_j\notin\Sigma$ such that

$$0<t_{j+1}<t_j/2^j, \quad ||w(t_j)||_{L^2(\Omega)}<1/j.$$

As j increases, we obtain the initial condition

$$\lim_{t\downarrow 0}||w(t)||_{L^2(\Omega)} = \varliminf_{j\to\infty}||w(t_j)||_{L^2(\Omega)}=0.$$

We go on to the observation of $w(t)\in H^1_0(\Omega)$ for almost all $t\in(0,T)$. The strong convergence (5.1) admits a subsequence of $\{w_{(h)}(t)\}_{h>0}$ such that

$$w_{(h)}(t,\cdot)\to w(t,\cdot) \ \text{ strongly in } L^2(\Omega) \ \text{ for a.a. } 0<t<T \tag{5.7}$$

as h tends to zero.

Fixing such t as makes (5.7) hold, we set a linear functional S_h of $H^1_0(\Omega)$ by

$$S_h(\varphi) := \int_\Omega (w_{(h)}(t)\cdot\varphi + Dw_{(h)}(t):D\varphi)\,dx. \tag{5.8}$$

Due to Schwarz's and Poincaré's inequalities, we obtain

$$\begin{aligned} |S_h(\varphi)| &\le ||w_{(h)}(t)||_{L^2(\Omega)}||\varphi||_{L^2(\Omega)} + ||Dw_{(h)}(t)||_{L^2(\Omega)}||D\varphi||_{L^2(\Omega)} \\ &\le C||Dw_{(h)}(t)||_{L^2(\Omega)}||D\varphi||_{L^2(\Omega)} + ||Dw_{(h)}(t)||_{L^2(\Omega)}||D\varphi||_{L^2(\Omega)} \le CK||\varphi||_{H^1_0(\Omega)}, \end{aligned}$$

for all $\varphi \in H_0^1(\Omega)$, where the constant K is taken from Lemma 1. Then there exists a subsequence of $\{h > 0\}$ such that $S_h(\varphi)$ has a limit $S(\varphi)$ in $\mathbb{R}$ for all $\varphi \in H_0^1(\Omega)$ as h goes to zero. Riesz's representation theorem captures a function $\overline{w} \in H_0^1(\Omega)$ which describes $S(\varphi)$ as

$$S(\varphi) = \int_\Omega (\overline{w} \cdot \varphi + D\overline{w} : D\varphi)\, dx. \tag{5.9}$$

On combination of (5.8) and (5.9), we get

$$w_{(h)}(t,\cdot) \rightharpoonup \overline{w}(t,\cdot) \ \text{ weakly in } H_0^1(\Omega) \ \text{ as } h \to 0, \tag{5.10}$$

and hence in particular (see [4])

$$w_{(h)}(t,\cdot) \rightharpoonup \overline{w}(t,\cdot) \ \text{ weakly in } L^2(\Omega) \ \text{ as } h \to 0.$$

Since the uniqueness of the weak limit in $L^2(\Omega)$ makes the two limits (5.7) and (5.10) coincide, We find $w(t,\cdot) \in H_0^1(\Omega)$ for almost all $0 < t < T$. Thus the lateral boundary condition is fulfilled and it is completely shown that w is a weak solution of (1.5).

The final task is to investigate the local Hölder continuity of w. We shall show that the Cauchy-Euler polygon $w^*_{(h)}$ has the same limit. Minkowski's inequality yields

$$\begin{aligned}
||w^*_{(h)} - w||_{L^2(Q_T)} &\le ||w^*_{(h)} - w_{(h)}||_{L^2(Q_T)} + ||w_{(h)} - w||_{L^2(Q_T)} \\
&= ||(t-t_n)\partial_t w_{(h)}||_{L^2(Q_T)} + ||w_{(h)} - w||_{L^2(Q_T)} \\
&< h||\partial_t w_{(h)}||_{L^2(Q_T)} + ||w_{(h)} - w||_{L^2(Q_T)}.
\end{aligned}$$

Due to (5.1) and (5.3), $w^*_{(h)}$ converges to w in $L^2(Q_T)$ as h tends to zero.

From (4.6), the sequence $\{w^*_{(h)}\}_{h>0}$ is equi-Hölder continuous in $\widetilde{Q}$ with exponent γ_0. By virtue of Ascoli-Arzelà's theorem, w is γ_0-Hölder continuous in $\widetilde{Q}$.

Recall that if w is a weak solution of (1.5), then $u = u_0 + e^{\mu t} w$ is a weak solution of (1.1). The assumption $Du_0 \in L^{2,m-2+2\gamma_0}_{\mathrm{loc}}(\Omega)$ is well known ([1]) to imply that u_0 belongs to the local Campanato space $\mathcal{L}^{2,m+2\gamma_0}_{\mathrm{loc}}(\Omega)$ consisting of a function f satisfying

$$\sup_{\substack{x_0 \in \widetilde{\Omega} \\ 0<r<\sigma}} r^{-(m+2\gamma_0)} \int_{B_r(x_0)} |f - (f)_{r,x_0}|^2\, dx < +\infty$$

for an arbitrary number $0 < \sigma < \operatorname{dist}(\widetilde{\Omega}, \partial\Omega)$ and an arbitrary subdomain $\widetilde{\Omega} \subset\subset \Omega$, which is isomorphic to $C^{\gamma_0}_{\mathrm{loc}}(\Omega)$ due to the classical Campanato imbedding theory ([2]). With attention to the boundedness of $w(t,x)$ in $\widetilde{Q}$ and the smoothness of $e^{\mu t}$ on $[\,0,T\,]$, we obtain

$$\begin{aligned}
|u(t,x) - u(t',x')| &\le |u_0(x) - u_0(x')| + |e^{\mu t} w(t,x) - e^{\mu t'} w(t',x')| \\
&\le C|x-x'|^{\gamma_0} + (e^{\mu t} - e^{\mu t'})|w(t,x)| + e^{\mu t'}|w(t,x) - w(t',x')| \\
&\le C|x-x'|^{\gamma_0} + C\mu|t-t'| + C(|t-t'|^{\gamma_0/2} + |x-x'|^{\gamma_0}) \\
&\le C(|t-t'|^{\gamma_0/2} + |x-x'|^{\gamma_0})
\end{aligned}$$

for (t,x), $(t',x') \in \widetilde{Q}$. Thus we come to the conclusion.

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