ON THE MAXIMAL NUMBER AND THE DIAMETER OF EXCEPTIONAL SURGERY SLOPE SETS

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ABSTRACT. Concerning the set of exceptional surgery slopes for a hyperbolic knot, Lackenby and Meyerhoff proved that the maximal cardinality is 10 and the maximal diameter is 8. Their proof is computer-aided in part, and both bounds are achieved simultaneously. In this note, it is observed that the diameter bound 8 implies the maximal cardinality bound 10 for exceptional surgery slope sets. This follows from the next known fact: For a hyperbolic knot, there exists a slope on the peripheral torus such that all exceptional surgery slopes have distance at most two from the slope. We also show that, in generic cases, the particular slope above can be taken as the slope represented by the shortest geodesic on a horotorus in a hyperbolic knot complement.

1. INTRODUCTION

In the study of 3-manifolds, one of the important operations describing the relationships between 3-manifolds would be *Dehn surgery*. We denote by $K(r)$ the resultant 3-manifold via Dehn surgery on a knot K along a slope r. (As usual, by a *slope*, we mean an isotopy class of a non-trivial unoriented simple loop on a torus.) Precisely, the 3-manifold $K(r)$ is obtained by removing an open tubular neighborhood $N(K)$ of K, and gluing a solid torus V back so that the slope r on the boundary torus of the complement of $N(K)$ is represented by the simple closed curve identified with the meridian of V.

As a consequence of the Geometrization Conjecture, raised by Thurston in [14, Conjecture 1.1], and established by celebrated Perelman's works [9, 10, 11], all closed orientable 3-manifolds are classified into four types: reducible, toroidal, Seifert fibered, and hyperbolic manifolds. Then we can observe that, generically, the structure of a knot complement persists in surgered manifolds. Actually the famous Hyperbolic Dehn Surgery Theorem, due to Thurston [13, Theorem 5.8.2], says that each hyperbolic knot (i.e., a knot with hyperbolic complement) admits only finitely many Dehn surgeries yielding non-hyperbolic manifolds. In view of this, such finitely many exceptions are called *exceptional surgeries*, and giving an interesting subject to study.

In this note, for a given hyperbolic knot K, $\mathcal{E}(K)$ denotes the set of the slopes along each of which the Dehn surgery on K is exceptional, and we call $\mathcal{E}(K)$ *the exceptional surgery slope set* for K.

This set $\mathcal{E}(K)$ is a finite set for each knot K, but as stated in [4, Problem 1.77(B)], Gordon conjectured that there exist the universal upper bounds on the cardinality and the diameter of such sets, which would be 10 and 8 respectively.

Here the diameter of $\mathcal{E}(K)$ is defined as the maximum of the distance (i.e., the minimal intersection number between their representatives) between a pair of the elements in $\mathcal{E}(K)$.

There had been many studies about the conjecture, and eventually, in [6], Lackenby and Meyerhoff gave an affirmative answer to the conjecture as follows.

Theorem ([6, Theorems 1.1 and 1.2]). *Let K be a hyperbolic knot in a closed orientable 3-manifold, and $\mathcal{E}(K)$ the exceptional surgery slope set for K. Then the cardinality of $\mathcal{E}(K)$ is at most 10, and the diameter of $\mathcal{E}(K)$ is at most 8.*

Their proof is computer-aided in part, and both bounds are achieved simultaneously. We claim in the next section that, for $\mathcal{E}(K)$, the diameter bound 8 actually implies the maximal cardinality bound 10. Note that such an implication can not hold for general sets of slopes, as remarked in [6, Section 2]. Actually there exists a set of slopes that its diameter is 8 but its cardinality is 12.

The key of our claim is the following known fact.

Proposition. *For any hyperbolic knot, there exists a slope on the peripheral torus such that all exceptional surgery slopes have distance at most two from the slope.*

This fact follows from [15, Theorem 2.5] and the unpublished result due to Gabai and Mosher together with an affirmative answer to Geometrization Conjecture. A part of the proof of Gabai-Mosher's theorem is included in the unpublished monograph [8]. See also [2, Theorem 6.48]. An independent proof for the theorem is also obtained by Calegari as a corollary of [2, Theorem 8.24].

Remark that both proofs by Gabai-Mosher and Calegari are very deep results based on the study of the foliation theory. In particular, the slope described in the proposition comes from an essential lamination in the knot exterior, called a *degeneracy slope*, and is rather difficult to compute in practice.

Concerning the proposition above, we show that, in generic cases, the slope represented by the shortest geodesic on a horotorus in a hyperbolic knot complement can play the same role as that particular slope in Proposition.

We here remark that, in [3], the author showed that, if for a hyperbolic knot K, there exists a slope on the peripheral torus such that all exceptional surgery slopes have distance at most ONE from the slope, then the cardinality of $\mathcal{E}(K)$ is at most 10, and the diameter of $\mathcal{E}(K)$ is at most 8.

2. Relationship between upper bounds

In this section, we give a proof of the following:

Theorem 1. *Let K be a hyperbolic knot in a closed orientable 3-manifold. Then, for $\mathcal{E}(K)$, if the diameter is at most 8, then the cardinality is at most 10.*

Proof. Recall first that it is well-known that slopes on a torus are parametrized by rational numbers with $1/0$, using a meridian-longitude system. See [12] for example.

Now, by virtue of Proposition, for K, there exists a slope γ on the peripheral torus such that all exceptional surgery slopes have distance at most two from γ. We set this γ to be the meridian, which corresponds to $1/0$. It should be noted that this γ can be an element of $\mathcal{E}(K)$.

We further set a longitude, which corresponds to $0/1$, and then we identify each element in $\mathcal{E}(K)$ other than γ as an irreducible fraction. Recall here that the distance between such a pair of slopes a/b and c/d is calculated as $|ad - bc|$.

Suppose first that there are no integral elements in $\mathcal{E}(K)$, equivalently, all the elements in $\mathcal{E}(K)$ have distance 2 from γ. Then any pair of elements, say $x/2$ and $y/2$ in $\mathcal{E}(K)$, other than γ has distance at least 4. Together with the assumption that the diameter of $\mathcal{E}(K)$ is at most 8, we see that the cardinality of $\mathcal{E}(K)$ is at most 4.

Thus we next suppose that $\mathcal{E}(K)$ contains some integral elements. In this case, after taking the mirror image if necessary, we can set a longitude such that integral elements in $\mathcal{E}(K)$ correspond to $\{0, \cdots, N_k\}$ with $N_i \geq 0$ for $1 \leq i \leq k$. Here k denotes the number of integral elements in $\mathcal{E}(K)$. We remark that $k \leq N_k + 1$ holds, and, since we are assuming that the diameter of $\mathcal{E}(K)$ is at most 8, $N_k \leq 8$ holds.

On the other hand, non-integral elements in $\mathcal{E}(K)$ are, if exist, all half integers, say $\{M_1/2, \cdots, M_l/2\}$ with M_j odd for $1 \leq j \leq l$. Then we have $|M_1 - 2N_k| \leq 8$ from the assumption that the diameter of $\mathcal{E}(K)$ is at most 8. It implies $-4 + N_k \leq M_1/2 \leq 4 + N_k$. Since M_1 is odd, we further obtain that $-7/2 + N_k \leq M_1/2 \leq 7/2 + N_k$. Similarly we have $-7/2 \leq M_l/2 \leq 7/2$, and then, it follows that $M_l/2 - M_1/2 \leq 7/2 - (-7/2 + N_k) = 7 - N_k$. This implies that the number of half-integral elements in $\mathcal{E}(K)$ is at most $8 - N_k$. Since $\mathcal{E}(K)$ consists of integral elements and half-integral elements together with γ, the cardinality of $\mathcal{E}(K)$ is at most $(N_k + 1) + (8 - N_k) + 1 = 10$. $\square$

As remarked in Section 1, for general sets of slopes, such a diameter bound does not imply the required cardinality bound. For example, we actually have the following set of slopes:

$$\left\{ \frac{1}{0}, \frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{3}, \frac{5}{4}, \frac{7}{4}, \frac{7}{5}, \frac{8}{5} \right\}$$

By direct calculations, we see that its diameter is 8, but its cardinality is 12.

3. Existence of particular slope

In this section, we give a proof of the following proposition.

Proposition 1. *If a hyperbolic knot complement contains a horotorus of area greater than $8/\sqrt{3}$, then the slope on the peripheral torus represented by the shortest geodesic on the horotorus have distance at most two from all exceptional surgery slopes for the knot.*

Proof. We recall some basic terminologies. Let K be a hyperbolic knot in a 3-manifold M. Then the universal cover of the complement C_K of K is identified with the hyperbolic 3-space $\mathbb{H}^3$. Under the covering projection, an equivariant set of horospheres bounding disjoint horoballs in $\mathbb{H}^3$ descends to a torus embedded in C_K, which we call a *horotorus*. As demonstrated in [13], a Euclidean metric on a horotorus T is obtained by restricting the hyperbolic metric of C_K. By using this metric, the length of a curve on T can be defined. Also T is naturally identified with the peripheral torus of K, since the image of the horoballs under the covering projection is topologically T times half open interval. Thus, for a slope r on the peripheral torus of K, we define the *length* of r with respect to T as the minimal length of the simple closed curves on T which represent the slope on T corresponding to the slope r.

Now suppose that T has the area greater than $8/\sqrt{3}$. Let γ be the shortest slope on T.

Claim. *If $\Delta(\gamma, \gamma') \geq 3$ holds for a slope γ', then the length of γ' is greater than 6.*

Proof. Let h be the length of γ, and w the length of the shortest path which starts and ends on γ, but which is not homotopic into γ. Then the length of a slope γ' is at least $w\Delta(\gamma, \gamma')$. Thus, to prove the claim, it suffices to show that $w > 2$.

Now we are supposing that the area of T is greater than $8/\sqrt{3}$, which is equal to wh. Then, in the case that $h \leq 4/\sqrt{3}$, we have $w > 8/\sqrt{3} \cdot 1/h \geq 2$ immediately.

On the other hand, in [7, Proof of Theorem, page 1049-1050], it is shown that $w \geq h\sqrt{3}/2$ holds in general. Thus, in the case that $h > 4/\sqrt{3}$, we have $w \geq h\sqrt{3}/2 > 2$.

These imply that the length of a slope γ' is greater than 6. $\square$

Finally we use the so-called "6-Theorem" due to Agol [1] and Lackenby [5] together with the affirmative answer to the Geometrization Conjecture, given by Perelman [9, 10, 11].

Claim. *Dehn surgery along a slope of length greater than 6 is non-exceptional.* $\square$

This completes the proof. $\square$

Acknowledgments

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