ALMOST SYMMETRIC NUMERICAL SEMIGROUPS OF
MULTIPLICITY 5

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ABSTRACT. In this paper, we study almost symmetric numerical semigroups of multiplicity 5. We give a classification of almost symmetric numerical semigroups of multiplicity 5 with embedding dimension 4.

1. INTRODUCTION

Throughout this paper, let \( \mathbb{N} \) denote the set of nonnegative integers. A numerical semigroup \( H \) is a subset of \( \mathbb{N} \) that is closed under addition, contains the zero element and has finite complement in \( \mathbb{N} \). A numerical semigroup \( H \) always admits a finite system of generator: that is, there exist integers \( n_1, ..., n_r \) such that 
\[
H = \langle n_1, ..., n_r \rangle := \{ \lambda_1 n_1 + \cdots + \lambda_r n_r \mid \lambda_1, ..., \lambda_r \in \mathbb{N} \}.
\]
Moreover, every numerical semigroup \( H \) has an unique minimal system of generators \( n_1, ..., n_b \), where the number \( b \) is called embedding dimension of \( H \). The least positive integer belonging to \( H \) is called the multiplicity of \( H \) and is denoted by \( e(H) \). In general, it holds that \( b \leq e(H) \). The Frobenius number of \( H \) is the maximum integer not belonging to \( H \), which is denoted by \( F(H) \). We always let the notation \( H = \langle n_1, ..., n_r \rangle \) denote a numerical semigroup minimally generating by \( \{ n_1, ..., n_r \} \), that is, \( r \) is the embedding dimension of \( H \).

For a numerical semigroup \( H = \langle n_1, ..., n_r \rangle \), we put \( k[H] := k[t^h \mid h \in H] \), where \( k \) is any fixed field and \( t \) a variable over \( k \). \( R = k[H] \) is called a numerical semigroup ring of \( H \) which has a graded ring structure the natural way (see [H]) and \( R \) has unique homogeneous maximal ideal \( m := (t^{n_1}, ..., t^{n_r}) \). It is known that \( a(R) = F(H) \), where \( a(R) \) is the a-invariant of \( k[H] \) (see [GW]). The multiplicity of \( R \) will be denoted by \( e(R) \). Then we have \( e(R) = e(H) \) and \( m \) denotes the number of minimal generating system for \( m \).

A numerical semigroup \( H \) is symmetric if for any integer \( x \in \mathbb{Z} \), either \( x \in H \) or \( F(H) - x \in H \). If \( H \) is symmetric, then \( F(H) \) must be odd by definition. A numerical semigroup \( H \) is pseudo-symmetric if \( F(H) \) is even and for any \( x \in \mathbb{Z} \)\((F(H)/2) \), either \( x \in H \) or \( F(H) - x \in H \). In [BF] the authors introduced a larger class of semigroups than symmetric or pseudo-symmetric. This class is called almost symmetric (see Definition 2.5). It is known that \( H \) is almost symmetric numerical semigroup if and only if \( k[H] \) is almost Gorenstein ring (see [BF]).

Let \( H = \langle n_1, ..., n_r \rangle \) be a numerical semigroup and \( S = k[X_1, ..., X_r] \) a polynomial ring in \( r \) variable over a field \( k \). Then we let \( I \) be the kernel of the homomorphism \( \phi : S \rightarrow k[H] \) of \( k \)-algebras defined by \( \phi(X_i) = t^{n_i} \) for each \( 0 \leq i \leq r \). Then \( I \) is
called the *defining ideal* of \( H \). We denote by \( \mu(I) \) the number of minimal generators of \( I \).

When the embedding dimension \( r = 3 \), J. Herzog in \([H]\) gave a complete characterization of the defining ideal \( I \), and he has proved that \( \mu(I) \leq 3 \). When \( r = 4 \) and the numerical semigroup \( H \) is symmetric, H. Bresinsky in \([B]\) gave a complete description of the defining ideal \( I \). In particular, he has proved that \( \mu(I) \leq 5 \).

The purpose of this paper is to make the complete list of numerical semigroups \( H = \langle 5, b, c, d \rangle \) which are almost symmetric. This is done in Theorem 3.3 and we see that \( \mu(I) = 5 \) if \( H \) is pseudo-symmetric and \( \mu(I) = 6 \) if \( H \) is almost symmetric with \( t(H) = 3 \).

2. PRELIMINARIES

In this section, we recall some definitions and basic facts about numerical semigroups and numerical semigroup rings.

**Definition 2.1.** Let \( H \) be a numerical semigroup.

1. We say that an integer \( x \) is a *pseudo-Frobenius number* if \( x \not\in H \) and \( x + h \in H \) for all \( h \in H \setminus \{0\} \). We denote by \( \text{PF}(H) \) the set of pseudo-Frobenius numbers of \( H \).

2. The cardinality of \( \text{PF}(H) \) is called the *type* of \( H \), denoted by \( t(H) \).

From the definition it easily follows that \( F(H) \in \text{PF}(H) \), in fact it is the maximum of \( \text{PF}(H) \).

**Proposition 2.2.** Let \( H \) be a numerical semigroup and \( R = k[H] \) its semigroup ring with unique homogeneous maximal ideal \( m \). Then

1. \( r(R) = t(H) \), where \( r(R) \) denote the Cohen-Macaulay type of \( R \).
2. \( a(R) = F(H) \).

**Proof.** Since the local cohomology module \( H^1_m(R) \) is generated by \( \{t^m \mid m \in \mathbb{Z} \setminus H\} \) as \( R \)-module, \( \{t^x \mid x \in \text{PF}(H)\} \) generates the socle of \( H^1_m(R) \). Hence we have \( r(R) = t(H) \) and \( a(R) = F(H) \).

The Apéry set is the key tool throughout this paper.

**Definition 2.3.** Let \( H \) be a numerical semigroup and \( n \) be one of its nonzero elements. The *Apéry set* of \( n \) in \( H \) is

\[
\text{Ap}(H, n) = \{ h \in H \mid h - n \not\in H \}.
\]

In another words, \( \text{Ap}(H, n) = \{0 = \omega(0), \omega(1), ..., \omega(n-1)\} \), where \( \omega(i) \) is the least element of \( H \) congruent \( i \) modulo \( n \), for all \( i \in \{0, ..., n-1\} \). It is also clear that \( F(H) = \max \text{Ap}(H, n) - n \) by the definition. More generally, we can get pseudo-Frobenius numbers of \( H \) from the Apéry set by the following way: Over the set of integers we define the relation \( \leq_H \), that is, \( a \leq_H b \) implies that \( b - a \in H \). Then we have the following result (see \([RG2]\)).

**Proposition 2.4.** Let \( H \) be a numerical semigroup and let \( n \) be a nonzero element of \( H \). Then

\[
\text{PF}(H) = \{ \omega - n \mid \omega \text{ is maximal with respect to } \leq_H \text{ in } \text{Ap}(H, n) \}.
\]
The concept of almost symmetric numerical semigroup was defined by Barucci and Fröberg in [BF].

**Definition 2.5.** [BF] We say that $H$ is an almost symmetric numerical semigroup if $z \not\in H$ implies that either $F(H) - z \in H$ or $z \in PF(H)$.

By Definition 2.5, a numerical semigroup $H$ is pseudo-symmetric if and only if $H$ is almost symmetric with $t(H) = 2$.

We set $PF(H) = \{f_1 < f_2 < \cdots < f_{t(H)} = F(H)\}$. In [N] it was proved that $H$ is almost symmetric if and only if $f_i + f_{t(H)-i} = F(H)$ for all $i \in \{1, 2, \ldots, t(H) - 1\}$.

**Theorem 2.6.** [N] Let $H$ be a numerical semigroup and let $n$ be one of its nonzero elements. Set $Ap(H, n) = \{0 < \alpha_1 < \cdots < \alpha_m\} \cup \{\beta_1 < \beta_2 < \cdots < \beta_{t(H)-1}\}$ with $m = n - t(H)$ and $PF(H) = \{\beta_i - n, \alpha_m - n = F(H) \mid 1 \leq i \leq t(H) - 1\}$. We put $f_i = \beta_i - n$ and $f_{t(H)} = \alpha_m - n = F(H)$. Then the following conditions are equivalent to each other.

1. $H$ is almost symmetric.
2. $\alpha_i + \alpha_{m-i} = \alpha_m$ for all $i \in \{1, 2, \ldots, m - 1\}$ and $\beta_j + \beta_{t(H) - j} = \alpha_m + n$ for all $j \in \{1, 2, \ldots, t(H) - 1\}$.
3. $f_i + f_{t(H)-i} = f_{t(H)}$ for all $i \in \{1, 2, \ldots, t(H) - 1\}$.

**Remark 2.7.** When $H$ is symmetric or pseudo-symmetric, the equivalence of (1) and (2) is shown in [RG2] Proposition 4.10 and 4.15.

By using Theorem 2.6, it is easy to show that a numerical semigroup $H$ is almost symmetric if and only if $2g(H) = F(H) + t(H)$ (see [BF]). Hence $H$ is almost symmetric with even-type (resp. odd-type) implies $F(H)$ is even integer (resp. odd integer).

Let $I$ be the defining ideal of $R = k[H]$. Namely $R = k[X_1, \ldots, X_r]/I$, of $H = \langle n_1, \ldots, n_r \rangle$.

**Proposition 2.8.** Let $H = \langle n_1, \ldots, n_r \rangle$ be a numerical semigroup and $I$ its defining ideal. Then $I$ is generated by homogeneous binomials

$$f(u, v) = \prod_{i=1}^{r} X_i^{u_i} - \prod_{i=1}^{r} X_i^{v_i},$$

where $u_i v_i = 0$ and $\deg f = \sum_{i=1}^{r} u_i n_i = \sum_{i=1}^{r} v_i n_i$.

Our goal is to study almost symmetric numerical semigroups of multiplicity 5 with embedding dimension 4. We have classified pseudo-symmetric numerical semigroups with embedding dimension 3 in [NNW]. Also, we show the classification of almost symmetric semigroups of multiplicity $\leq 4$ in the following Remark. Then we will have the classification of all almost symmetric semigroups of multiplicity $\leq 5$.

**Remark 2.9.** Let $H = \langle a, b, c, d \rangle$ be a numerical semigroup with multiplicity $a$ and generated by 4 elements. Then $a \geq 4$ and if $a = 4$, then $t(H) = 3$ and $Ap(H, 4) = \{0, b, c, d\}$. Using Theorem 2.6, it is easy to see that $H$ is almost symmetric if and only if after change of variables, $b = 2\alpha + \beta + 1$, $c = 2\beta + 2$, $d = 2\alpha + 3\beta - 1$, where $\alpha$ (resp. $\beta$) is any positive (resp. even positive) integer.
3. Main Theorem

Let $H = \langle 5, b, c, d \rangle$ be a numerical semigroup of multiplicity 5. First we show that $t(H) \leq 3$.

**Lemma 3.1.** Let $H = \langle 5, b, c, d \rangle$ be a numerical semigroup of multiplicity 5. Then $t(H) \leq 3$.

Moreover, if $t(H) = 3$ and if we put $\text{Ap}(H, 5) = \{0, b, c, d, \omega\}$, then $\omega = 2d$ after rearranging $\{b, c, d\}$.

**Proof.** Set $\text{Ap}(H, 5) = \{0, b, c, d, \omega\}$. Since $\omega$ is not a minimal generator of $H$, one of $b, c, d \leq_H \omega$ in the notation of Proposition 2.4. Hence at least one element among $\{b, c, d, \omega\}$ is not maximal with respect to $\leq_H$ and $t(H) \leq 3$ by Proposition 2.4.

If $t(H) = 3$, then only one element among $\{b, c, d, \omega\}$, say, $d$ is $\leq_H \omega$. Hence $\omega = nd$ for some $n \geq 2$. If $n \geq 3$, then $b$ or $c \leq_H 2d \leq_H \omega$. Hence $\omega = 2d$. □

The following lemma characterizes almost symmetric numerical semigroups of multiplicity 5 with embedding dimension 4.

**Lemma 3.2.** Let $H = \langle 5, b, c, d \rangle$ be a numerical semigroup of multiplicity 5. Then

1. $H$ is pseudo-symmetric if and only if $5 + b + c = 2d$ after permuting $b, c$ and $d$ if necessary.
2. $H$ is almost symmetric with $t(H) = 3$ if and only if $b + c = 2d + 5$ after permuting $b, c$ and $d$ if necessary.
3. $H$ is symmetric if and only if $b + c = 2d$ if necessary.

**Proof.** The statements (1) and (3) are Proposition 4.10 and 4.15 of [RG2]. But we reproduce the proof for the convenience of readers. We put $\text{Ap}(H, 5) = \{0, b, c, d, \omega\}$ throughout the proof.

(1) Assume $\text{PF}(H) = \{d - 5, F(H)\}$ with $F(H) = 2(d - 5)$. Then, since $b - 5, c - 5 \leq_H F(H)$ by Proposition 2.4. Hence we must have $F(H) = \omega - 5 = 2(d - 5)$ and $\omega = b + c$.

(2) If $t(H) = 3$, by Lemma 3.1 we may assume that $\omega = F(H) + 5 = 2d$ and $\text{PF}(H) = \{b - 5, c - 5, 2d - 5\}$. We see from Theorem 2.6 that $H$ is almost symmetric if and only if $b + c = 2d + 5$.

(3) Again by Proposition 2.4, we must have $b, c, d \leq_H F(H) + 5$ and hence $\omega = F(H) + 5$. □

Now we can prove our main theorem.

**Theorem 3.3.** Let $H = \langle 5, b, c, d \rangle$ be a numerical semigroup of multiplicity 5. Then after suitable permutation of $\{b, c, d\}$, we have the following expressions.

1. $H$ is pseudo-symmetric if and only if

\[
\begin{align*}
b &= 3\alpha + 2\beta + 1, \\
c &= \alpha + 4\beta + 2, \\
d &= 2\alpha + 3\beta + 4,
\end{align*}
\]

where $\alpha, \beta \geq 1$ with $\beta - \alpha \not\equiv 2 \pmod{5}$.

In this case $F(H) = 4\alpha + 6\beta - 2$. 

(2) \( H \) is almost symmetric with \( t(H) = 3 \) if and only if
\[
\begin{align*}
b &= 3\alpha + 2\beta - 1, \quad c = \alpha + 4\beta - 2, \quad d = 2\alpha + 3\beta - 4, \\
\text{where } &\alpha \geq 1 \text{ and } \beta \geq 2 \text{ with } \beta - \alpha \not\equiv 3 \pmod{5}.
\end{align*}
\]
In this case \( F(H) = 4\alpha + 6\beta - 13. \)

(3) \( H \) is symmetric if and only if
\[
\begin{align*}
b &= 3\alpha + 2\beta, \quad c = \alpha + 4\beta, \quad d = 2\alpha + 3\beta, \\
\text{where } &\alpha, \beta \geq 1 \text{ with } \alpha \not\equiv \beta \pmod{5}.
\end{align*}
\]
In this case \( F(H) = 4\alpha + 6\beta - 5. \)

Proof. (1) As in Lemma 3.2, put \( \text{Ap}(H, 5) = \{0, b, c, d, \omega\} \) and assume that \( b + d \equiv 0 \pmod{5} \) and \( c + \omega \equiv c + 2d \equiv 0 \pmod{5} \). Then \( 2b \equiv c, 2c \equiv d \) and \( c + d \equiv b \pmod{5} \).

Put \( 2b = c + 5\alpha \) and \( 2c = d + 5\beta \) with \( \alpha, \beta \geq 1 \) and recall that \( 2d = b + c + 5 \). Then we get \( b = 3\alpha + 2\beta + 1, c = \alpha + 4\beta + 2, d = 2\alpha + 3\beta + 4 \) and \( F(H) = 2(d - 5) = 4\alpha + 6\beta - 2. \) Note that \( b, c, d \equiv 0 \pmod{5} \) if and only if \( \alpha - \beta \equiv 3 \) and otherwise, \( 5, b, c, d \) are relatively prime and \( \langle 5, b, c, d \rangle \) is a numerical semigroup.

(2) As in (1), choose \( b, c \) so that \( b + d \equiv 0 \pmod{5} \) and \( c + \omega \equiv c + 2d \equiv 0 \pmod{5} \) as in (1) and put \( 2b = c + 5\alpha \) and \( 2c = d + 5\beta \) as well. Then since \( b + c = 2d + 5 \) in this case, we have
\[
\begin{align*}
b &= 3\alpha + 2\beta - 1, \quad c = \alpha + 4\beta - 2, \quad d = 2\alpha + 3\beta - 4 \quad \text{and} \quad F(H) = 2d - 5 = 4\alpha + 6\beta - 13.
\end{align*}
\]
Note that \( b, c, d \equiv 0 \pmod{5} \) if and only if \( \alpha - \beta \equiv 2 \pmod{5} \) and otherwise, \( 5, b, c, d \) are relatively prime and \( \langle 5, b, c, d \rangle \) is a numerical semigroup.

(3) As in (1), (2), choose \( b, c \) so that \( b + d \equiv 0 \pmod{5} \) and \( c + \omega \equiv c + 2d \equiv 0 \pmod{5} \) as in (1) and put \( 2b = c + 5\alpha \) and \( 2c = d + 5\beta \) as well. Then since \( b + c = 2d + 5 \) in this case, we have
\[
\begin{align*}
b &= 3\alpha + 2\beta, \quad c = \alpha + 4\beta, \quad d = 2\alpha + 3\beta \quad \text{and} \quad F(H) = 2d - 5 = 4\alpha + 6\beta - 5.
\end{align*}
\]
Note that \( b, c, d \equiv 0 \pmod{5} \) if and only if \( \alpha \equiv \beta \pmod{5} \) and otherwise, \( 5, b, c, d \) are relatively prime and \( \langle 5, b, c, d \rangle \) is a numerical semigroup.

We can give the complete description of the defining ideal of an almost symmetric numerical semigroup of multiplicity 5 with embedding dimension 4.

**Corollary 3.4.** Let \( H = \langle 5, b, c, d \rangle \) be a numerical semigroup of multiplicity 5 and \( I \) its defining ideal. Then

(1) \( H \) is pseudo-symmetric if and only after permuting variables, if necessary, \( I \) is the form of
\[
I = (X^{\alpha + \beta + 1} - YW, Y^2 - X^\alpha Z, Z^2 - X^\beta W, W^2 - XYZ, X^{\beta + 1}Y - ZW),
\]
where \( \alpha, \beta \geq 1 \) and \( \beta - \alpha \not\equiv 2 \pmod{5} \).

(2) \( H \) is almost symmetric with \( t(H) = 3 \) if and only after permuting variables, if necessary, \( I \) is the form of
\[
I = (X^{\alpha + \beta} - YW, Y^3 - X^\alpha W^2, Z^2 - X^{\alpha + 1}Y, W^2 - X^\beta Z, X^\alpha W - YZ, XY^2 - ZW),
\]
where \( \alpha \geq 1 \) and \( \beta \geq 2 \) with \( \beta - \alpha \not\equiv 3 \pmod{5} \).
(3) \(H\) is symmetric if and only if after permuting variables, if necessary, \(I\) is the form of
\[
I = (X^{\alpha + \beta} - ZW, Y^2 - X^\alpha Z, Z^2 - YW, W^2 - X^\beta Y, X^\alpha W - YZ),
\]
where \(\alpha, \beta \geq 1\) with \(\alpha \neq \beta \pmod{5}\).

**Proof.** Let \(I\) be the ideals described in (3.4) – (3.6). It is clear from Theorem 3.3 that \(I\) is contained in the defining ideal of \(k[H]\) in each case. Also, we can check that \(\dim_k k[X, Y, Z, W]/(I, X) = 5\), which implies that \(I\) is the defining ideal of \(H\).

**Corollary 3.5.** Let \(H = \langle 5, b, c, d \rangle\) be a numerical semigroup of multiplicity 5 and \(I\) be its defining ideal.

1. If \(H\) is pseudo-symmetric, then \(\mu(I) = 5\).
2. If \(H\) is almost symmetric with \(t(H) = 3\), then \(\mu(I) = 6\).
3. If \(H\) is symmetric, then \(\mu(I) = 5\).

**Question 3.6.** Are the results in 3.5 always true for almost symmetric numerical semigroups generated by 4 elements? Namely, assume that \(H = \langle a, b, c, d \rangle\) be an almost symmetric numerical semigroup generated by 4 elements.

1. Is \(t(H) \leq 3\) always?
2. Does \(\mu(I) = 5\) holds if \(H\) is pseudo-symmetric?
3. Does \(\mu(I) = 6\) holds if \(t(H) = 3\)?

**Remark 3.7.** The results in Corollary 3.5 (1) and (3) follow from Theorem 3.8 of by J. C. Rosales and P. A. García-Sánchez and Theorem 3.9 by H. Bresinsky, respectively, since \(H = \langle 5, b, c, d \rangle\) is not a complete intersection. (If \(k[H]\) is a complete intersection, then \(e(H) \geq 8\).

**Theorem 3.8.** [RG1] Let \(H = \langle n_1, ..., n_r \rangle\) be a numerical semigroup such that \(r = e(H) - 1 = n_1 - 1\) and \(I\) its defining ideal. Let \(Ap(H, n_1) = \{0, n_2, ..., n_r, \omega\}\).

1. If \(\omega = n_i + n_j\) for some \(i, j \in \{2, ..., r\}, i \neq j\), then
\[
\mu(I) = \frac{(n_1 - 1)(n_1 - 2)}{2} - 1.
\]
2. Otherwise,
\[
\mu(I) = \frac{(n_1 - 1)(n_1 - 2)}{2}.
\]

**Theorem 3.9.** [B] Let \(H = \langle n_1, n_2, n_3, n_4 \rangle\) be symmetric and \(I\) its defining ideal. Then \(I\) is not a complete intersection if and only if
\[
I = (X_1^{\alpha_1} - X_3^{\alpha_3} X_4^{\alpha_4}, X_2^{\alpha_2} - X_1^{\alpha_1} X_4^{\alpha_4}, X_3^{\alpha_3} - X_1^{\alpha_1} X_2^{\alpha_2}, X_4^{\alpha_4} - X_2^{\alpha_2} X_3^{\alpha_3}, X_3^{\alpha_3} X_1^{\alpha_1} - X_2^{\alpha_2} X_4^{\alpha_4}),
\]
where, each of generators for \(I\) is unique up to isomorphism and \(0 < \alpha_{ij} < \alpha_j\) for each \(i, j\).

We can construct almost symmetric numerical semigroups \(H = \langle 5, b, c, d \rangle\) with the given (possible) Frobenius number.

**Theorem 3.10.** Let \(f\) be a positive integer which is not a multiple of 5.
(1) There exists a pseudo-symmetric numerical semigroup $H = \langle 5, b, c, d \rangle$ with $F(H) = f$ if $f$ is even and $f \geq 8$.

(2) There exists an almost symmetric numerical semigroup $H = \langle 5, b, c, d \rangle$ with $t(H) = 3$ and $F(H) = f$ if $f$ is odd and $f \geq 7$.

(3) There exists a symmetric numerical semigroup $H = \langle 5, b, c, d \rangle$ with $F(H) = f$ if $f$ is odd and $f \geq 9$.

Proof. This follows easily from Theorem 3.3. \qed

Example 3.11. (1) Let $H = \langle 5, 12, 19, 18 \rangle$. Then the defining ideal of $H$ is

$$I = (X^6 - YW, Y^2 - XZ, Z^2 - X^4W, W^2 - XYZ, X^3Y - ZW),$$

and we can see that $H$ is pseudo-symmetric with $F(H) = 26$. In fact, we get $H$ putting $\alpha = 1, \beta = 4$ in Theorem 3.3 (1).

(2) Let $H = \langle 5, b, c, d \rangle$ be pseudo-symmetric numerical semigroup with $F(H) = 26$. Then by solving the equation $4\alpha + 6\beta - 2 = 26$ in Theorem 3.3 (1), we find that either $(\alpha, \beta) = (1, 4)$ or $(4, 2)$. Hence $H = \langle 5, 12, 19, 18 \rangle$ as above or $H = \langle 5, 17, 14, 18 \rangle$.

(3) Let $H = \langle 5, 11, 13, 14 \rangle$. Then the defining ideal of $H$ is

$$I = (X^5 - YW, Y^3 - XW^2, Z^2 - X^3Y, W^2 - X^3Z, X^2W - YZ, XY^2 - ZW).$$

Hence $H$ is almost symmetric with $t(H) = 3$ and $F(H) = 17$ by (3.5).

(4) Assume $H = \langle 5, b, c, d \rangle$ be an almost symmetric numerical semigroup with $t(H) = 3$ with $F(H) = 17$. Then solving $4\alpha + 6\beta - 13 = 17$, we get $(\alpha, \beta) = (3, 2)$ and we see that $H = \langle 5, 11, 13, 14 \rangle$ is the unique one with $t(H) = 3$ and $F(H) = 17$.

(5) Let $H = \langle 5, 18, 16, 14 \rangle$. Then its defining ideal is

$$I = (X^5 - ZW, Y^2 - X^4Z, Z^2 - YW, W^2 - X^2Y, X^4W - YZ).$$

This implies $H$ is symmetric with $F(H) = 27$ by Theorem 3.3 (3).

(6) If $H = \langle 5, 19, 13, 16 \rangle$ is symmetric with $F(H) = 27$, then solving $4\alpha + 6\beta - 5 = 27$, we get $(\alpha, \beta) = (5, 2)$ or $(2, 4)$. Then we see that $H = \langle 5, 19, 13, 16 \rangle$ and $H = \langle 5, 14, 18, 16 \rangle$ are the symmetric numerical semigroups satisfying this condition.

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