Adjoint Functors attached to Group Homomorphisms
and the van Kampen Theorem

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Abstract. The aim of this note is to provide two proofs for the basic fact about existence of adjoint functors attached to group homomorphisms. The first proof is straightforward and known. Another is given as an application of general category theory and somewhat conceptional. We supply the argument of [5] in detail.

Introduction. Let \( f : G \to G' \) be a group homomorphism. \( f \) induces a functor \( f^* \) from the category of \( G' \)-sets to the category of \( G \)-sets. Then \( f^* \) admits adjoint functors. These functors are basic and fundamental in cohomology or representation theories of groups. The purpose of this note is to explain how one can apply the elementary facts of category theory to obtain these adjoints in a primitive situation. We give a complete proof for a characterization of group isomorphisms via \( f^* \) and apply it to prove the van Kampen theorem. This paper is expository in nature.

Adjoint

(1.0). Let \( G \) be a group. We denote by \( 1_G \) or 1 the identity element of \( G \). By a \( G \)-set, we mean a set \( S \) with a left \( G \) action. For clarity of expression, if \( g, h \in G \) and \( s \in S \), we use the notation \( g \cdot s \) for the group action of \( g \) to \( s \) and \( g h \) for group multiplication of \( g \) and \( h \). Let \( S, T \) be two \( G \)-sets. A \( G \)-map \( u : S \to T \) is a map such that

\[ u(g \cdot s) = g \cdot u(s) \quad \text{for all } g \in G \text{ and } s \in S. \]

We denote by \( \text{Hom}_G(S, T) \) the set of all \( G \)-maps from \( S \) to \( T \). The category whose objects are \( G \)-sets and whose morphisms are \( G \)-maps is called the category of \( G \)-sets and denoted by \( G \text{-Set} \). (In other literature, this is denoted by \( B_G \) and called the classifying space or, more generally, classifying topos.)

Let \( G, G' \) be groups and let \( f : G \to G' \) be a group homomorphism. If \( G' \) acts on a set \( S' \), then \( f \) induces an action of \( G \) on \( S' \) in an obvious way; \( g \cdot s' := f(g) \cdot s' \) for \( g \in G, s \in S' \). We denote this \( G \)-set \( S' \) by \( f^*(S') \). It is obvious that \( f \) induces a functor from the category of
$G'$-sets to the category of $G$-sets:

$$f^* : G'\text{-}\text{Set} \rightarrow G\text{-}\text{Set}, \quad S' \mapsto f^*(S').$$

On the other hand, for a $G$-set $S$, $G'$ acts on the set $\text{Hom}_G(f^*(G'), S)$ by letting

$$(g' \cdot u)(h') := u(h'g') \quad \text{for} \quad u \in \text{Hom}_G(f^*(G'), S) \quad \text{and} \quad g', h' \in G'.$$

It is easy to see that this is indeed an action of $G'$. This $G'$-set is denoted by $f_*(S)$. If $K$ is the kernel of $f$, then for every $u \in f_*(S)$ the image of $u$ is contained in the subset $S^K := \{ s \in S \mid g \cdot s = s, \forall g \in K \}$ of $K$ invariant elements of $S$. Indeed, for every $g \in K$ and every $g' \in G' = f^*(G')$

$$g \cdot u(g') = u(f(g)g') = u(1_{G'}g') = u(g').$$

Thus

$$f_*(S) := \text{Hom}_G(f^*(G'), S) = \text{Hom}_G(G', S^K).$$

For $G$-sets $S$, $T$ and $a \in \text{Hom}_G(S, T)$, we define

$$(f_*(a))(u) := a \circ u \quad \text{for} \quad u \in f_*(S).$$

Then $f_*(a) : f_*(S) \rightarrow f_*(T)$ is a $G'$-map. Indeed, for $g', h' \in G'$ and $u \in f_*(S)$

$$(f_*(a)(g' \cdot u))(h') = (a \circ (g' \cdot u))(h')$$
$$= a(g' \cdot u(h'))$$
$$= a(u((h'g')))$$
$$= (a \circ u)(h'g')$$
$$= (f_*(a)(u))(h'g')$$
$$= (g' \cdot (f_*(a)(u)))(h').$$

It follows from this that $f_*$ defines a functor

$$f_* : G\text{-}\text{Set} \rightarrow G'\text{-}\text{Set}, \quad S \mapsto f_*(S).$$

Remark. If $f : G \rightarrow G'$ is a surjective homomorphism (resp. an injective homomorphism whose image is a normal subgroup) of groups and $S$ is a $G$-module then $f_*$ is known as the inflation map (resp. induced map) in the cohomology theory of groups. Of course, in our case the algebraic structures on the sets on which groups act or topologies on groups are not regarded for simplifying the consideration.

Proposition (1.1). Let $f : G \rightarrow G'$ and $f' : G' \rightarrow G''$ be two group homomorphisms. Then we have a canonical isomorphism of
functors

\[(f'f)_* \simeq f'_* f_* .\]

It follows that there is a covariant lax-functor from the category of groups to the category of sets\(^1\):

**Group \(\rightarrow\) Cat \((G \mapsto \text{-Set})\).**

*Proof.* Let \(S\) be a \(G\)-set. Define a map \(\varphi : f'_*(f_*(S)) \rightarrow (f'f)_*(S)\) by letting for \(v \in f'_*(f_*(S)) = \text{Hom}_G(G'', \text{Hom}_G(G', S))\) and \(g'' \in G''\)

\[\varphi(v)(g'') = v(g'')(1_{G''})\]

We will show that \(\varphi\) is an isomorphism (bijection).

**Injectivity of \(\varphi\):**
Suppose that \(\varphi(v) = \varphi(w)\) for \(v, w \in f'_*(f_*(S))\). Then

\[(v(g''))(g') = (g' \cdot v(g''))(1_{G'}) \quad \text{(by the definition of } G' \text{ action)}
\]
\[= v(f'(g')g'')(1_{G'}) \quad \text{(} v \text{ is a } G' \text{-map)}
\]
\[= \varphi(v)(f'(g')g'') \quad \text{(by the definition of } \varphi)\]
\[= \varphi(w)(f'(g')g'')\]
\[= (w(g''))(g') \quad \text{for all } g' \in G', g'' \in G''.\]

Hence

\[v = w.\]

Thus \(\varphi\) is an injection.

**Surjectivity of \(\varphi\):**
For \(u \in \text{Hom}_G(G'', S)\), let \(\psi(u) : G'' \rightarrow \text{Hom}_G(G', S)\) be the map defined by

\[(\psi(u)(g''))(g') = u(f'(g')g'') \quad \text{for } g' \in G', g'' \in G''.\]

Then \(\psi(u) \in f'_*(f_*(S))\), that is, \(\psi(u)\) is a \(G'\)-map. In fact, for every \(g', h' \in G'\) and every \(g'' \in G''\)

\[(g' \cdot (\psi(u)(g''))(h') = (\psi(u)(g''))(h'g')
\]
\[= u(f'(h'g')g'')
\]
\[= u(f'(h')f'(g')g'')
\]
\[= (\psi(u)(f'(g')g''))(h').\]

The composition of \(\psi\) and \(\varphi\) satisfies

\[(\varphi(\psi(u)))(g'') = (\psi(u)(g''))(1_{G}) = u(g'') \quad \text{for all } g'' \in G''.\]

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\(^1\)To avoid the set-theoretical difficulty, we fix a universe in the sense of (SGA 1, I Appendix) and work in this set.
It follows that
\[ \varphi(\psi(u)) = u. \]
This shows that \( \varphi \) is a surjection.

(1.2). Let \( f : G \rightarrow G' \) be a group homomorphism. \( f \) factors into the composite of the canonical surjective homomorphism \( \rho : G \rightarrow f(G) \) followed by the inclusion \( i : f(G) \rightarrow G' \). Then we have canonical identification
\[ f_* = i_* \rho_* . \]

(1.3). Let \( f : G \rightarrow G' \) be a group homomorphism. There exist two natural transformations of functors
\[ \alpha : 1_{G'\text{-Set}} \rightarrow f_* f^*, \quad \beta : f^* f_* \rightarrow 1_{G\text{-Set}} . \]
These are defined by the following ways:
For a \( G'\text{-set} S' \) let \( \alpha_{S'} : S' \rightarrow f_*(f^*(S')) \) be the map
\[ (\alpha_{S'}(s'))(g') := g' \cdot s' \quad \text{for} \quad s' \in S' \text{ and } g' \in G' \]
and for a \( G\text{-set} S \) let \( \beta_S : f^*(f_*(S)) \rightarrow S \) be the map
\[ \beta_S(u) := u(1_{G'}) \quad \text{for} \quad u \in f^*(f_*(S)) = \text{Hom}_G(G', S) . \]

**Proposition (1.4).** There is an isomorphism
\[ \text{Hom}_G(f^*(S'), S) \simeq \text{Hom}_{G'}(S', f_*(S)) \]
given by the correspondences
\[ u \mapsto f_*(u) \circ \alpha_{S'} \quad \text{and} \quad v \mapsto \beta_S \circ f^*(v) \]
for \( u \in \text{Hom}_G(f^*(S'), S) \) and \( v \in \text{Hom}_{G'}(S', f_*(S)) \). This is functorial in \( S \) and \( S' \).
In other words, \((f^*, f_*, \alpha, \beta)\) is an adjoint system of functors between \( G\text{-Set} \) and \( G'\text{-Set} \).

**Proof.** We define two maps
\[ \zeta : \text{Hom}_G(f^*(S'), S) \rightarrow \text{Hom}_{G'}(S', f_*(S)) \]
and
\[ \xi : \text{Hom}_{G'}(S', f_*(S)) \rightarrow \text{Hom}_G(f^*(S'), S) \]
by the following rules:
For \( a \in \text{Hom}_G(f^*(S'), S) \) and \( b \in \text{Hom}_{G'}(S', f_*(S)) \),

\[
(\zeta(a))(s') = f_*(a)(\alpha_{S'}(s')) = a \circ (\alpha_{S'}(s'))
\]

and

\[
(\xi(b))(s') = (\beta_S \circ b)(s') = (b(s'))(1_{G'}) \quad \text{for } s' \in S'.
\]

Hence we have

\[
((\xi \circ \zeta)(a))(s') = (\zeta(a)(s'))(1_{G'}) = (a \circ \alpha_{S'}(s'))(1_{G'}) = a(s')
\]

for every \( a \in \text{Hom}_G(f^*(S'), S) \) and \( s' \in S' \).

Thus

\[
\xi \circ \zeta = \text{id}.
\]

Similarly, for every \( b \in \text{Hom}_{G'}(S', f_*(S)) \)

\[
((\zeta \circ \xi)(b))(s') = (f_*(\xi(b)) \circ \alpha_{S'})(s') = \xi(b) \circ \alpha_{S'}(s') = b(s') \quad \text{for all } s' \in S'.
\]

So

\[
\zeta \circ \xi = \text{id}.
\]

Since the above arguments are obviously functorial with respect to \( S \)
and \( S' \), we complete the proof.

(1.5). We wish to give a conceptional explanation for (1.3) and
(1.4) by using the notions and basic facts in category theory. This
gives us somewhat clear way for understanding about adjoint functors
which we are concerned with.

Let \( G \) be a group. \( G \) defines a category \( \mathcal{C}_G \) whose set of objects
consists of a single element and whose set of morphisms is the group
\( G \). When we consider elements \( g, h \in G \) as morphisms of \( \mathcal{C}_G \),
the composition of morphisms \( g \) followed by \( h \) is defined to be group
multiplication \( hg \in G \). If \( f : G \to G' \) is a group homomorphism, then
it defines a functor from \( \mathcal{C}_G \) to \( \mathcal{C}_{G'} \) in an obvious way. We denote this
functor by the same letter \( f \). Let \( \textbf{Set} \) be the category of sets. To every
\( G \)-set \( S \) one can define a functor \( F_S \) from \( \mathcal{C}_G \) to \( \textbf{Set} \) which associates
the unique object of \( \mathcal{C}_G \) to the set \( S \) and each element \( g \) of \( G \) to a
bijection(automorphism) of \( S \) given by the action of \( g \). Conversely, for
every functor \( F \) from \( \mathcal{C}_G \) to \( \textbf{Set} \) we have a \( G \)-set to which the unique
object of \( \mathcal{C}_G \) corresponds by the functor \( F \). By these correspondences,
we may identify the category \( G \text{-}\textbf{Set} \) with the category \( \text{Funct}(\mathcal{C}_G, \textbf{Set}) \)
of functors from \( \mathcal{C}_G \) to \( \textbf{Set} \):

\[
G \text{-}\textbf{Set} \simeq \text{Funct}(\mathcal{C}_G, \textbf{Set})).
\]
For a $G'$-set $S'$, by composition of the two functors $f : C_G \to C_{G'}$ and $F_{S'} : C_{G'} \to \textbf{Set}$ we have a functor

$$F_{S'} \circ f : C_G \to \textbf{Set}.$$  

The $G$-set corresponding to this functor is $f^*(S')$.

The following fact is known as the notion of Kan extension of a functor and yields a fundamental method to construct (left or right) adjoint functors.

**Proposition (1.6).** Let $\mathcal{C}, \mathcal{C'}$ be categories and $f : \mathcal{C} \to \mathcal{C'}$ be a functor. $f$ induces the functor by composition

$$f^* : \text{Funct} (\mathcal{C'}, \text{Set}) \to \text{Funct} (\mathcal{C}, \text{Set}) \quad (F \mapsto F \circ f).$$

Then $f^*$ admits a left adjoint functor $f_!$ and a right adjoint functor $f_*$, that is, these are functors from $\text{Funct} (\mathcal{C}, \text{Set})$ to $\text{Funct} (\mathcal{C'}, \text{Set})$ such that there exist isomorphisms

$$\text{Hom}_{\text{Funct} (\mathcal{C'}, \text{Set})} (f_! (E), F) \simeq \text{Hom}_{\text{Funct} (\mathcal{C}, \text{Set})} (E, f^*(F)),$$

$$\text{Hom}_{\text{Funct} (\mathcal{C}, \text{Set})} (f^*(E), F) \simeq \text{Hom}_{\text{Funct} (\mathcal{C'}, \text{Set})} (F, f_* (E))$$

functorial for $E \in \text{ob}(\text{Funct} (\mathcal{C}, \text{Set}))$ and $F \in \text{ob}(\text{Funct} (\mathcal{C'}, \text{Set}))$.

**Proof.** We refer to ([4] SGA 4, I, 5), ([5], 2) for detailed verifications. The construction procedure for the functor $f_!$ and $f_*$ is as follows.

1st step: We have to define $f_! (E)(X')$ for each object $X'$ of $\mathcal{C'}$. Let us define the category $\mathcal{I}_f^{X'}$ whose objects are all pairs $(X, m')$ where $X \in \text{ob}(\mathcal{C})$ and $m' : f(X) \to X'$ is a morphism in $\mathcal{C'}$. The morphisms $m : (X_1, m'_1) \to (X_2, m'_2)$ are the morphisms $m : X_1 \to X_2$ in $\mathcal{C}$ such that $m'_1 = m'_2 \circ f(m)$.

![Diagram](\text{Insert Diagram Here})

Taking the inductive limit of the family $\{(E(X), E(m)) | (X, m) \in \text{ob}(\mathcal{I}_f^{X'})\}$, we define

$$f_! (E)(X') := \lim_{(X, m) \in \text{ob}(\mathcal{I}_f^{X'})} E(X).$$
For a morphism $m' : X'_1 \to X'_2$ in $\mathcal{C}'$, $m'$ induces a functor from $\mathcal{I}_f^{X'_1}$ to $\mathcal{I}_f^{X'_2}$ in the obvious way. Then, by the universal property of inductive limit, there exists a map

$$f_1(E)(X'_1) \to f_1(E)(X'_2).$$

It is easy to see that $f_1(E)$ is a covariant functor from $\mathcal{C}'$ to $\textbf{Set}$.

2$^{\text{nd}}$ step: We show that there exist isomorphisms

$$\text{Hom}_{\text{Func}(\mathcal{C}', \textbf{Set})}(f_1(E), F) \xrightarrow{\sigma} \text{Hom}_{\text{Func}(\mathcal{C}, \textbf{Set})}(E, f^*(F))$$

functorial in both $E$ and $F$.

For every $X \in \text{ob}(\mathcal{C})$, $(X, \text{id}_{f(X)})$ is an object of $\text{ob}(\mathcal{I}_f^{f(X)})$. By definition of the inductive limit, there exists a canonical map

$$E(X) \to f_1(E)(f(X)).$$

For $u \in \text{Hom}_{\text{Func}(\mathcal{C}', \textbf{Set})}(f_1E, F)$ let us consider the composite map

$$\sigma(u)(X) := [E(X) \to f_1(E)(f(X)) \xrightarrow{u(f(X))} F(f(X)) = f^*(F)(X)].$$

This is functorial for $X$. Thus we have a morphism of functors

$$\sigma(u) : E \to f^*(F).$$

Conversely, let $v \in \text{Hom}_{\text{Func}(\mathcal{C}, \textbf{Set})}(E, f^*(F))$. For an object $X'$ of $\text{ob}(\mathcal{C})$, if $(X, m') \in \text{ob}(\mathcal{I}_f^{X'})$ then we have a composite map

$$[E(X) \xrightarrow{v(X)} f^*(F)(X) = F(f(X)) \xrightarrow{F(m')} F(X')].$$

By passing to the inductive limit, it induces a canonical map

$$\tau(v)(X') : f_1(E)(X') \to F(X').$$

This is functorial for $X'$. Hence we have a morphism

$$\tau(v) : f_1(E) \to F.$$

It is easy to prove that these two morphisms are inverse to each others.

Let us apply the above construction to the opposite categories $\mathcal{C}^o$, $\mathcal{C}'^o$, $\textbf{Set}^o$ and the functor $f^o$ and use the canonical isomorphism of categories

$$\text{Func}(\mathcal{C}, \textbf{Set}) \simeq \text{Func}(\mathcal{C}^o, \textbf{Set}^o)^o.$$
$$(X_1, m'_1) \to (X_2, m'_2)$$ are the morphisms $m : X_1 \to X_2$ in $\mathcal{C}$ such that $m'_2 = f(m) \circ m'_1$.

By taking the projective limit, we define

$$f_*(E)(X') := \lim_{(X, m) \in \text{ob}(\mathcal{I}'_{X'})} E(X).$$

A morphism $m' : X'_1 \to X'_2$ induces a functor $\mathcal{I}'_{X'_2} \to \mathcal{I}'_{X'_1}$. By passing to the projective limit, we have a morphism $(f_*(E))(X'_1) \to (f_*(E))(X'_2)$. It is easy to verify that $f_* : \text{Funct}(\mathcal{C}, \text{Set}) \to \text{Funct}(\mathcal{C}', \text{Set})$ is a functor.

(1.7). We will examine the above procedure for constructing adjoint functors in our situation (1.5).

We denote by $\bullet$ (resp. $\bullet'$) the unique object of the category $\mathcal{C}_G$ (resp. $\mathcal{C}_{G'}$). Then the category $\mathcal{I}'_{\bullet'} = \mathcal{I}'_\bullet$ which we have to consider is described as follows:

The set of objects of $\mathcal{I}'_\bullet$ (resp. $\mathcal{I}'_{\bullet'}$) is canonically identified with $G'$, and for objects $g'$, $h'$ of $\mathcal{I}'_\bullet$, the morphisms from $g'$ to $h'$ are the elements $g \in G$ such that $g' = h'f(g)$:

$$f(\bullet) = \bullet' \xrightarrow{f(g)} f(\bullet) = \bullet'$$

So we have

$$\text{ob}(\mathcal{I}'_\bullet) = G',$$

$$\text{Hom}_{\mathcal{I}'_\bullet}(g', h') = \begin{cases} \{ gK \} & \text{if } g' = h'f(g) \\ \emptyset & \text{otherwise} \end{cases}$$

where $K$ is the kernel of $f$.

Similarly, the category $\mathcal{I}'_{\bullet'}$ is as follows:

$$\text{ob}(\mathcal{I}'_{\bullet'}) = G',$$

$$\text{Hom}_{\mathcal{I}'_{\bullet'}}(g', h') = \begin{cases} \{ gK \} & \text{if } h' = f(g)g' \\ \emptyset & \text{otherwise} \end{cases}$$
By the above observations, if \( \{ g_i' \}_{i \in I} \) is a complete set of left representatives of \( G'/H' \) where \( H' \) is the image of \( f \), we have the following expressions for \( f_!(S) \):
\[
f_!(S) = f_!(F_S)(\bullet') = \lim_{(\bullet, g) \in \text{ob}(\tau')} F_S(\bullet) = \prod_{i \in I} (S/K)_i
\]
where \( (S/K)_i \) are copies of the quotient set of \( S \) by the action of \( K \) for each \( i \in I \).

For an element \( s \in S \) we denote \( [s] \) the equivalence class of \( s \) modulo \( K \) and \( [s]_i \) the element \( [s] \) considered as an element of \( (S/K)_i \). With these notations, the action of \( G' \) on \( \prod_{i \in I} (S/K)_i \) is given by the following rule: for \( g' \in G' \), if \( g'g'_i = g'_j f(g) \) for some \( g \in G \) then
\[
g' \cdot [s]_i := [g \cdot s]_j \quad \text{for} \quad [s]_i \in \prod_{i \in I} (S/K)_i.
\]
Since \( K \) is a normal subgroup of \( G \), this is well-defined.

Similarly, expression for \( f_*(S) \) is as follows:
\[
f_*(S) = f_*(F_S)(\bullet') = \lim_{(\bullet, g) \in \text{ob}(\tau')} F_S(\bullet) = \prod_{i \in I} S^K_i
\]
where \( S^K_i \) are copies of \( S^K \) of \( K \) invariant elements of \( S \) for each \( i \in I \). The action of \( G' \) on \( \prod_{i \in I} S^K_i \) is given by the following rule: for \( g' \in G' \) and \( < s > := (s_i)_{i \in I} \) with \( s_i \in S^K_i \), if \( g^{-1}_j g' = f(g)g^{-1}_i \) for some \( g \in G \) then \( j \)-th component of \( g' \cdot < s > \in \prod_{i \in I} S^K_i \) is defined to the element
\[
g' \cdot < s > := g \cdot s_i.
\]
Since \( s_i \) is a \( K \) invariant element of \( S \), this is independent of the choice of \( g \).

\textbf{(1.8).} Let \( f : G \rightarrow G' \) and \( f' : G' \rightarrow G'' \) be two group homomorphisms. Then we have a canonical isomorphism of functors
\[
(f' \circ f)_! \simeq f'_! \circ f_!.
\]
Proof. There are isomorphisms functorial for a $G$-set $E$ and $G'$-set $F$:

$$\text{Hom}_{G''}(f'_!(f_!(E)), F) \cong \text{Hom}_{G'}(f_!(E), f'^*(F))$$
$$\cong \text{Hom}_G(E, f^*f'^*(F))$$
$$\cong \text{Hom}_G(E, (f' f)^*((F)))$$
$$\cong \text{Hom}_{G''}((f' f)_!(E), F).$$

It follows that there is a canonical isomorphism of functors $(f' f)_! \cong f'_! f_!$.

(1.9). Let $G_i (i = 1, 2)$ be groups and $S_i (i = 1, 2)$ be a $G_i$-set. Let $G = G_1 \times G_2$ be the direct product of groups and $p_i : G \longrightarrow G_i (i = 1, 2)$ be the projection. Then

$$\text{Hom}_{G_1}(p_{1!} p_2^*(S_2), S_1) \cong \text{Hom}_{G_2}(S_2, p_2^* p_1^*(S_1)).$$

We give a characterization of group isomorphisms.

Proposition (1.10). Let $f : G \longrightarrow G'$ be a group homomorphisms. If $f^*$ is an equivalence of categories, then $f$ is an isomorphism.

Proof. We know that if $f^*$ is an equivalence then adjoint morphism of $f^*$ are canonically isomorphic. Moreover, $f^*$ is an equivalence if and only if the adjunction morphisms

$$\alpha : 1_{G'} \text{-Set} \cong f_* f^* \quad \text{and} \quad \beta : f^* f_* \cong 1_{G} \text{-Set}$$

are both isomorphisms ([6, IV,1 cor.1, 4 Thm. 1]).

Let $H'$ be the image of $f$, then as $G'$-sets

$$G'/H' \cong f_*(f^*(G'/H')) = \text{Hom}_G(G', G'/H').$$

Let $c : G' \longrightarrow G'/H'$ be the constant map defined by $c(g') = H'$ for all $g' \in G'$. Then $g \cdot (c(g')) = g \cdot H' = f(g)H' = H' = c(g \cdot g')$ for all $g \in G$ and $g' \in G'$. So $c$ is a $G'$-map, that is, $c \in \text{Hom}_{G'}(G', G'/H')$. Moreover, $c$ is a fixed point under the action of $G'$. In fact, by the definition of $G'$-action on $\text{Hom}_{G'}(G', G'/H')$, we have

$$(g' \cdot c)(h') = c(h'g') = H = c(h') \quad \text{for all} \quad g' \text{ and } h' \in G'.$$

So $G'$ acts on $G'/H'$ transitively with a fixed point. This implies that $G'/H'$ consists of a single element. Hence we have $G' = H'$. This shows that $f$ is a surjection. It remains to show that $f$ is an injection.

Let $K$ be the kernel of $f$ and $\beta = \beta_G : f^*(f_*(G)) \longrightarrow G$ be the adjunction morphism. Since $\beta$ is a $G$-map, it follows that

$$g \cdot (\beta(m)) = \beta(g \cdot m) = \beta(f(g) \cdot m) = \beta(1_{G'} \cdot m) = \beta(m)$$
for all \( g \in K \) and all \( m \in f^*(f_*'(G)) = \text{Hom}_G(G', G) \). As \( \beta \) is an isomorphism of \( G \)-sets by our assumption, this shows that \( K \) acts trivially on \( G \). Since the action of \( K \) is merely the one defined by group multiplication of \( G \), this shows that \( K \) is trivial.

**van Kampen Theorem**

For the basic properties of fundamental groups we refer to [1], [2], [3] and [8].

\[(2.0)\] Let \( X \) be a connected and locally simply connected topological space with base point \( x_0 \). We mean a path in \( X \) to be a continuous mapping from the unit interval to \( X \); \( c : [0, 1] \rightarrow X \) and denote by \([c]\) the homotopy class of \( c \). Let \( P(X) \) be the set of homotopy classes of paths starting at \( x_0 \). If \([c] \in P(X)\) and \( U \) is an open neighborhood of \( c(1) \) in \( X \), we denote \([c, U]\) the subset of \( P(X) \) consisting of all homotopy classes of composite paths of the form \( c \cdot c' \) where \( c' \) is a path in \( U \) starting at \( c(1) \). Then the collection \([\{[c, U]\}]\) is a base of a topology of \( P(X) \), and so determines a topology on \( P(X) \). We consider \( P(X) \) to be a topological space with respect to this topology. If \( p_X : P(X) \rightarrow X \) is a mapping defined by \( p_X([c]) := c(1) \), then \( p_X \) is a continuous mapping satisfying the following property; for every point of \( X \) there is an open neighborhood \( V \) such that \( p_X^{-1}(V) \) is a disjoint union of open set each of which is homeomorphic to \( V \) by \( p \). The fiber \( p_X^{-1}(x_0) := \{[c] \in P(X) | p_X([c]) = x_0\} \) of \( p_X \) over \( x_0 \) is a group under the multiplication law induced by composition of loops at \( x_0 \). This group is called the fundamental group of \( X \) and denoted by \( \pi_1(X, x_0) \). Similarly, the composition of paths induce an action of \( \pi_1(X, x_0) \) on \( P(X) \) over \( X \).

Let \( p : \tilde{X} \rightarrow X \) be a continuous mapping of topological spaces. If there is a homeomorphism \( f : \tilde{X} \rightarrow P(X) \) such that \( p = p_X \circ f \), then we say that \( p : \tilde{X} \rightarrow X \), or simply \( \tilde{X} \), is a universal covering of \( X \). \( \pi_1(X, x_0) \) acts on every universal covering of \( X \) over \( X \) in an obvious way. We can show that \( \pi_1(X, x_0) \), is canonically isomorphic to the group of automorphisms \( \text{Aut}_X(\tilde{X}) \) of \( \tilde{X} \) over \( X \) (i.e. transformation group of \( \tilde{X} \) over \( X \)).

\[(2.0.1)\] The groupoid of \( X \) is the category \( Grp_X \) whose set of objects is \( X \) and whose set of morphisms, \( \text{Hom}_{Grp_X}(x, y) \), is the set of all homotopy classes of paths from \( y \) to \( x \) for every \( x, y \in X \). A local system on \( X \) with values in a category of sets is a covariant functor from \( Grp_X \) to \( \text{Set} \).
Here is a basic example of local system on $X$:

We define $L_x(x)$ to be the fundamental group $\pi_1(X, x)$ of $X$ with base point $x$ for each $x \in X$ and define $L_x([c]) : \pi_1(X, x) \longrightarrow \pi_1(X, y)$ to be the group isomorphism determined by the homotopy class $[c] \in \text{Hom}_{\text{Grp}_X}(x, y)$ where $c : [0, 1] \longrightarrow X$ is a path with $y = c(0)$, $x = c(1)$ for $x, y \in X$. Then $L_x$ is a local system, called the local system of fundamental groups on $X$.

(2.0.2). There are equivalences among the three categories:

\[
\begin{align*}
\longrightarrow & \text{The category of locally constant sheaves on } X \\
\longrightarrow & \text{The category of } \pi_1(X, x_0)-\text{sets} \\
\longrightarrow & \text{The category of local systems on } X
\end{align*}
\]

We shall describe the above correspondences for equivalences.

(2.0.2.0). (1) If $L$ is a local system of sets on $X$, we can associate $L$ with a locally constant sheaf $\mathcal{F}_L$ on $X$. The set of sections of $\mathcal{F}_L$ over an open set $U$ of $X$ is defined to be the set

\[
\mathcal{F}_L(U) := \{ \sigma : U \longrightarrow \coprod_{x \in U} L(x) \mid \sigma(x) \in L(x) \text{ for all } x \in U \}
\]

and $L([c])(\sigma(c(1))) = \sigma(c(0))$ for all paths $c$ in $X$.

It is easy to see that this is a sheaf and, indeed locally constant because $X$ is locally simply connected.

(2.0.2.1). (2) If $\mathcal{F}$ is a locally constant sheaf of sets on $X$ then for any path $c : [0, 1] \longrightarrow X$, the inverse image $c^*(\mathcal{F})$ is a constant sheaf on the unit interval $[0, 1]$ because the unit interval is contractible to a point. So we have an isomorphism

\[
\iota_c : c^*(\mathcal{F})_1 = \mathcal{F}_{c(1)} \overset{\cong}{\longrightarrow} c^*(\mathcal{F})_0 = \mathcal{F}_{c(0)}
\]

which is determined by the homotopy class $[c]$. It is easy to see that the correspondence

\[
L_{\mathcal{F}} : \text{Grp}_X \longrightarrow \text{Set}, \quad L_{\mathcal{F}}(x) := \mathcal{F}_x, \quad L_{\mathcal{F}}([c]) := \iota_c
\]

is a local system on $X$. 

(24)
The above argument shows that \( \pi_1(X, x) \) acts on the fiber \( \mathcal{F}_x \) for every \( x \in X \), in particular we obtain a \( \pi_1(X, x_0) \)-set \( \mathcal{F}_{x_0} \).

**2.0.2.2.** (3) Let \( S \) be a \( \pi_1(X, x_0) \)-set. \( \pi_1(X, x_0) \) acts on the set \( P(X) \times S \) over \( X \) by

\[
\mu : \pi_1(X, x_0) \times (P(X) \times S) \rightarrow P(X) \times S,
\]

\[
(\alpha, ([c], s)) \mapsto (\alpha \cdot [c], \alpha \cdot s)
\]

for \( \alpha \in \pi_1(X, x_0) \) and \([c] \in P(X)\). By passing to the quotient under the group action, we have a mapping

\[
p : P(X) \times S / \pi_1(X, x_0) \rightarrow X.
\]

For each \( x \in X \), if we fix a point in the fiber \( p^{-1}_X(x) \), we obtain an isomorphism from the fiber \( p^{-1}_X(x) \) to \( S \). In particular, the identity element of \( \pi_1(X, x_0) = p^{-1}_X(x_0) \) gives an identification \( p^{-1}(x_0) \) with \( S \). If \([c] \in \text{Hom}_{\text{Grp}_X}(x, y)\) is a morphism represented by a path in \( X \) such that \( c(0) = y \) and \( c(1) = x \), then the path \( c^{-1} \), defined by \( c^{-1}(t) := c(1 - t) \) for \( t \in [0, 1] \), induces a map

\[
\iota_c : p^{-1}_X(x) \rightarrow p^{-1}_X(y)
\]

by composite of paths again; for \([a] \in p^{-1}_X(x)\) represented by a path \( a \) starting at \( x_0 \) and ending at \( x \), we associate \([a] \) with the homotopy class of composite path \( a \) followed by \( c^{-1} \). This is a \( \pi_1(X, x_0) \) equivariant map. Thus, by taking the quotient we have a map

\[
\iota_c : p^{-1}(x) \rightarrow p^{-1}(y).
\]

\( \iota_c \) depends only on the homotopy class of the path \( c \).

The correspondence \( L_S : \text{Grp}_X \rightarrow \text{Set} \) defined by

\[
L_S(x) := p^{-1}(x), \quad L_S([c]) := \iota_{[c]}
\]

for all \( x \in X \) and \([c] \in \text{Hom}_{\text{Grp}_X}(x, y)\)

is indeed a local system on \( X \).

The locally constant sheaf \( \mathcal{F}_{L_S} \) associated with \( \pi_1(X, x_0) \)-set \( S \) has \( S \) as its fiber over \( x_0 \).

**2.0.3.** Let \( S \) be a \( \pi_1(X, x_0) \)-set. By (2.0.2.0) and (2.0.2.2), we see that the sections of \( \mathcal{F}_S \) over an open set \( U \) of \( X \) are the equivariant locally constant maps from \( p^{-1}_X(U) \) to \( S \):

\[
\mathcal{F}_S(U) := \{ f : p^{-1}_X(U) \rightarrow S \mid f \text{ locally constant map such that } \ f(\alpha \cdot \tilde{x}) = \alpha \cdot f(\tilde{x}) \text{ for all } \tilde{x} \in p^{-1}_X(U) \text{ and } \alpha \in \pi_1(X) \}.
\]

Next, we shall describe this set by the fundamental group of \( U \).
Lemma (2.1). Let $U \subseteq X$ be a connected open subset of $X$ and $u_0 \in U$. Let $Z$ be a connected component of $p_X^{-1}(U)$ and $z_0 \in Z$ be a point such that $u_0 = p_X(z_0)$. Put $G_Z = \{ \alpha \in \pi_1(X) \mid \alpha(Z) \subseteq Z \}$. Then we have the followings:

(1) $G_Z = \{ \alpha \in \pi_1(X) \mid \alpha(Z) = Z \}$, so this is a subgroup of $\pi_1(X)$.

(2) The automorphism group $\text{Aut}_U(Z)$ of $Z$ over $U$ is canonically isomorphic to $G_Z$, and we have isomorphisms of groups:

$$G_Z \simeq \text{Aut}_U(Z) \simeq \pi_1(U, u_0)/\pi_1(Z, z_0)$$

where we identify $\pi_1(Z, z_0)$ with its image by the homomorphism induced by $p_X|_Z$ which is injective as $p_X|_Z$ is a covering morphism.

(3) If $j_{U, X} : \pi_1(U, u_0) \longrightarrow \pi_1(X, x_0)$ is the composite homomorphism $\pi_1(U, u_0) \longrightarrow \pi_1(X, u_0)$ induced by the inclusion and an isomorphism obtained by choosing a path $c$ from $x_0$ to $u_0$ in $X$, then the kernel of $j_{U, X}$ is canonically isomorphic to $\pi_1(Z, z_0)$:

$$\ker(j_{U, X}) \simeq \pi_1(Z, z_0).$$

Proof. To prove (1), it is sufficient to note the following equivalence which come from the facts that $\alpha \in \pi_1(X)$ is an automorphism of $\tilde{X}$ over $X$ and $Z$ is a connected component of $p_X^{-1}(U)$:

(a) $\alpha(Z) = Z$,

(b) $\alpha(Z) \subseteq Z$,

(c) $\alpha(z) \in Z$ for some point $z \in Z$.

Let $\sigma$ be an element of $\text{Aut}_U(Z)$. Then $\sigma(z_0) = \alpha(z_0)$ for some $\alpha \in \pi_1(X)$, as $\pi_1(X)$ acts transitively on the fiber of $p_X$ over $u_0$. By the above equivalence we have $\alpha \in G_Z$. Since $p_X|_Z : Z \longrightarrow U$ is a local homeomorphism and $p \circ \sigma = p = p \circ \alpha$, it follows that $\sigma$ and $\alpha$ coincide on an open neighborhood of $z_0$. This implies that the subset of $Z$ on which $\sigma$ and $\alpha$ agree is a non empty open set. Similar argument shows that the complement of this set is open. As $Z$ is a connected set, it follows that $\sigma = \alpha$. This proves that $G_Z$ is isomorphic to $\text{Aut}_U(Z)$. Since $G_Z$ acts on the fiber $p_X|_Z^{-1}(u_0)$ transitively, so does $\text{Aut}_U(Z)$. This implies that $\pi_1(Z, z_0)$ is a normal subgroup of $\pi_1(U, u_0)$ and $\text{Aut}_U(Z)$ is canonically isomorphic to the quotient group $\pi_1(U, u_0)/\pi_1(Z, z_0)$.

To prove (3), it suffices to note that the homomorphism $j_{U, X}$ is equal to the composition of the following homomorphisms:

$$\pi_1(U, u_0) \longrightarrow \pi_1(U, u_0)/\pi_1(Z, z_0) \xrightarrow{\simeq} G_Z \xrightarrow{\subset} \pi_1(X, x_0).$$
Lemma (2.2). Under the same notations and conditions as in (2.0) and (2.1), we have two isomorphisms

\[ \rho : \mathcal{F}_S(U) \longrightarrow S^{G_Z} \]

and

\[ \omega_c : S^{G_Z} \longrightarrow (j_{U,X}^*(S))_{\pi_1(U, u_0)} \]

where \( S^{G_Z} \) is the set of \( G_Z \)-invariant elements of \( S \) and the same for \( j_{U,X}^*(S) \). \( \omega_c \) depends on the homotopy class of path \( c \).

**Proof.** If \( f \) is an element of \( \mathcal{F}_S(U) \), then \( {}^\iota f \), the restriction of \( f \) on \( Z \), is a constant map because \( Z \) is connected. So, \( {}^\iota f \) is identified with an element of \( S \). The condition required for \( f \) in (2.0.3) implies that this element lies in \( S^{G_Z} \). It follows that we have a map

\[ \rho : \mathcal{F}_S(U) \longrightarrow S^{G_Z}, \quad \rho(f) := {}^\iota f. \]

Let \( \{ \alpha_i \}_{i \in I} \) be a complete set of left representatives of \( \pi_1(X)/G_Z \). Then we have

\[ p_X^{-1}(U) = \coprod_{i \in I} \alpha_i(Z) \quad \text{(disjoint union)}. \]

For a constant map \( g_s \) on \( Z \) with value \( s \), we define a map

\[ {}^*g_s : p_X^{-1}(U) \longrightarrow S \]

by

\[ {}^*g_s(\tilde{x}) := \alpha_i \cdot s \quad \text{for} \quad \tilde{x} \in \alpha_i(Z). \]

Then it is obvious that \( {}^*g_s \) is a locally constant map on \( p_X^{-1}(U) \) with \( {}^\iota({}^*g_s) = g_s \). We show that \( {}^*g_s \) is indeed an element of \( \mathcal{F}_S(U) \). For \( \tilde{x} \in \alpha_i(Z) \) and \( \alpha \in \pi_1(X) \), if \( \alpha \cdot \tilde{x} \in \alpha_j(Z) \) then, as \( \alpha_j^{-1} \alpha \alpha_i \in G_Z \)

\[
{}^*g_s(\alpha \cdot \tilde{x}) = \alpha_j \cdot s \\
= \alpha_j \cdot (\alpha_j^{-1} \alpha \alpha_i \cdot s) \\
= \alpha \cdot (\alpha_i \cdot \tilde{x}) \\
= \alpha \cdot {}^*g_s(\tilde{x}).
\]

This shows that \( \rho \) is a bijection.

By the isomorphism in ((2.1), (3)), it is obvious that \( S^{G_Z} \) is isomorphic to \( (j_{U,X}^*(S))_{\pi_1(U, u_0)} \).

**Proposition (2.3).** Let \( X \) be a connected and locally simply connected topological space with base point \( x_0 \). Let \( S \) be a \( \pi_1(X, x_0) \)-set. If \( W \) is a connected open subset of \( X \) containing \( x_0 \), then there exists a canonical isomorphism of sheaves on \( W \):

\[ \zeta_{w,x} = \zeta_w : i_w^*(\mathcal{F}_S) \overset{\cong}{\longrightarrow} \mathcal{F}_{i_w^*(S)} \]
such that if $W_2 \subseteq W_1$ are connected open subsets of $X$ containing $x_0$ then the following diagram is commutative:

$$
\begin{array}{ccc}
i_2^*(\mathcal{F}_S) = i_{2,1}^*(i_1^*(\mathcal{F}_S)) & \xrightarrow{i_{2,1}^*(\zeta_1)} & i_2^*(\mathcal{F}_{i_1^*(S)}) \\
\zeta_2 & \downarrow & \zeta_{2,1}
\end{array}
$$

where $i_{2,1} : W_2 \rightarrow W_1$, $i_1 : W_1 \rightarrow X$ and $i_2 : W_2 \rightarrow X$ are inclusions, and $\zeta_l = \zeta_{W_l} (l = 1, 2)$, $\zeta_{2,1} = \zeta_{W_2, W_1}$. (We have used the symbol $i_w$ for two different meanings, the first is an inclusion of topological space and the second is the induced group homomorphism.)

**Proof.** If $U$ is an open set of $W$. Let $u_0$ be a point of $U$ and let $c$ be a path from $x_0$ to $u_0$ in $W$. $c$ induces a homomorphism $j_{U, X} : \pi_1(U, u_0) \rightarrow \pi_1(X, x_0)$ as in (2.1). This homomorphism is equal to the composite homomorphism

$$
j_{U, X} : \pi_1(U, u_0) \xrightarrow{j_{U, W}} \pi_1(W, w_0) \xrightarrow{i_{w, X}} \pi_1(X, x_0).
$$

So $j_{U, X}^*(S) = j_{U, W}^*((i_{w, X})^*(S))$. Then, by (2.2) we have the following isomorphisms

$$
\begin{align*}
i_w^*(\mathcal{F}_S)(U) &= \mathcal{F}_S(U) \quad \rightarrow \quad (j_{U, X}^*)^\pi_1(U, u_0) \quad \text{(isomorphism in (2.2))} \\
j_{U, W}^*((i_w, X)^*(S)) &\pi_1(U, u_0) \quad \rightarrow \quad \mathcal{F}_w^*(S)(U) \quad \text{(inverse of (2.2)).}
\end{align*}
$$

We define $\zeta_W(U)$ to be the composition of the above isomorphisms. Then the family $\{ \zeta_W(U) \mid U \text{ open set of } X \}$ gives the desired isomorphism $\zeta_W$ of sheaves. We note that $\zeta_W(U)$ does not depend on a choice of path $c$. By tracing the definition of $\zeta_W$, it is easy to see that the following equality holds:

$$
\zeta_2 = \zeta_{2,1} \circ i_{2,1}^*(\zeta_1).
$$

**Theorem (2.4)** (van Kampen). Let $X$ be a connected and locally simply connected topological space and let $\{ W_\lambda \}_{\lambda \in \Lambda}$ be a family of connected open subsets of $X$ such that the intersection of any two of these open subsets is in this family. If $X = \bigcup_{\lambda \in \Lambda} W_\lambda$, and there is a point $x_0 \in \cap_{\lambda \in \Lambda} W_\lambda$, then the natural homomorphism of groups

$$
f : \lim_{\Lambda} \pi_1(W_\lambda, x_0) \rightarrow \pi_1(X, x_0)
$$

is an isomorphism.
Proof. ([7]). We denote \( \pi_1(W_\lambda, x_0) \) by \( G_\lambda \). The groups \( G_\lambda \) and the homomorphisms \( i_{\lambda, \mu} : G_\mu \to G_\lambda \) induced by the inclusion \( i_{\lambda, \mu} : W_\mu \to W_\lambda \) form an inductive system. We denote by \( G \) the inductive limit \( \lim_{\lambda \in \Lambda} \pi_1(W_\lambda, x_0) \). The homomorphisms \( i_\lambda : G_\lambda \to \pi_1(X, x_0) \) induced by the inclusions \( i_\lambda : W_\lambda \to X \) (\( \lambda \in \Lambda \)) determine a homomorphism \( f : G \to \pi_1(X, x_0) \). By (1.10), to prove the assertion it suffices to show that the functor

\[ f^* : \pi_1(X, x_0) \text{-} \text{Set} \to G \text{-} \text{Set} \]

is an equivalence of categories. Let \( t_\lambda : G_\lambda \to G \) be the canonical homomorphism, then \( i_\lambda = f \circ t_\lambda \) (\( \lambda \in \Lambda \)).

Let \( T \) be a set. To give an action of \( G \) on \( S \) is the same as to give a system of actions of \( G_\lambda \) on \( S \) for each \( \lambda \in \Lambda \) which are compatible with the group homomorphisms \( i_{\lambda, \mu} \). Let \( T \) be a \( G \)-set. We denote by \( T_\lambda \) the \( G_\lambda \)-set \( t_\lambda(T) \). We have a locally constant sheaf \( F_{T_\lambda} \) on the open set \( W_\lambda \) for each \( \lambda \in \Lambda \) and the canonical isomorphisms \( \zeta_{\lambda, \mu} : i_{\lambda, \mu}^* F_{T_\lambda} \to F_{i_{\lambda, \mu}^*(T_\lambda)} = F_{T_\mu} \) by (2.3). The family of those locally constant sheaves and isomorphisms determine the locally constant sheaf \( G_T \) on \( X \) such that \( i_\lambda^*(G_T) = F_{T_\lambda} \). In fact, for every open set \( U \) of \( X \) we define

\[ G_T(U) := \lim_{\lambda \in \Lambda} F_{T_\lambda}(U_\lambda) \]

where \( U_\lambda = U \cap W_\lambda \). Then by the assumption that the family of open sets is closed under finite intersection and using (2.3), we can easily verify that \( G_T \) is indeed the sheaf having required property. Taking the fiber of \( G_T \) over \( x_0 \) we have a \( \pi_1(X, x_0) \)-set \( G_{T, x_0} \). We consider the \( G \)-set \( f^*(G_{T, x_0}) \). This \( G \)-set is determined by the family of \( G_\lambda \)-sets \( t_\lambda^*(f^*(G_{T, x_0})) \). But we have

\[ t_\lambda^*(f^*(G_{T, x_0})) = i_\lambda^*(G_{T, x_0}) = (i_\lambda^* G_T)_{x_0} = (F_{T_\lambda})_{x_0} = T \]

by (2.3) again. Hence \( f^*(G_{T, x_0}) = T \). This shows that \( f^* \) is an essentially surjective functor. If \( T = f^*(S) \) for a \( \pi_1(X, x_0) \)-set \( S \), then \( F_{T_\lambda} = i_\lambda^* F_S \) for every \( \lambda \in \Lambda \) because \( T_\lambda = i_\lambda^*(S) \). It follows that

\[ G_T = F_S. \]

Let \( S_1, S_2 \) be two \( \pi_1(X, x_0) \)-sets and \( m \in \text{Hom}_{G \text{-} \text{Set}}(f^*(S_1), f^*(S_2)) \) be a \( G \)-map. By the similar argument given above, it is easy to verify that \( m \) determines the morphism of locally constant sheaves \( \mu : F_{S_1} \to F_{S_2} \) such that \( f^*(\mu_{x_0}) = m \) where \( \mu_{x_0} : (F_{S_1})_{x_0} = S_1 \to (F_{S_2})_{x_0} = S_2 \). This shows that \( f^* \) is a full functor. Faithfulness of the functor is obvious. This shows that \( f^* \) is an equivalence, which concludes the proof.
Remark. Two notions of locally constant sheaf and covering space are interchangeable. So, our proof of (2.4) is essentially the same as those in [2, 14c] and [3, X].

References


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