PSEUDO-SYMMETRIC AND ATOMIC NUMERICAL SEMIGROU...G, GENERATED BY THREE ELEMENTS

HIROKATSU NARI, TAKAHIRO NUMATA, AND KEI-ICHI WATANABE

(Received October 31, 2010)

ABSTRACT. In this paper we study numerical semigroups generated by 3 elements. We give a characterization of pseudo-symmetric semigroups. Also, we study construction of an atomic semigroup which can be obtained by deleting one element from symmetric or pseudo-symmetric semigroup.

1. Introduction

Let \( \mathbb{N} \) be the set of nonnegative integers. A numerical semigroup \( H \) is a subset of \( \mathbb{N} \) which is closed under addition and \( \mathbb{N} \setminus H \) is a finite set. We always assume \( 0 \in H \).

For a fixed field \( k \), let \( t \) be a variable over \( k \). Then we put \( k[H] = k[t^h \mid h \in H] \) and call it the semigroup ring of \( H \) over \( k \). Note that \( k[H] \) is a subring of \( k[t] = k[\mathbb{N}] \). Every numerical semigroup \( H \) admits a finite system of generators, that is, there exist \( a_1, \ldots, a_n \in S \) such that \( H = \langle a_1, \ldots, a_n \rangle = \{ \lambda_1 a_1 + \cdots + \lambda_n a_n \mid \lambda_1, \ldots, \lambda_n \in \mathbb{N} \} \).

Moreover, every numerical semigroup admits a unique minimal system of generators.

Let \( H \) be a numerical semigroup and let \( \{ a_1 < a_2 < \cdots < a_n \} \) be its minimal system of generators. We call \( a_1 \) the multiplicity of \( H \) and denote it by \( m(H) \), and we call \( n \) the embedding dimension of \( H \) and denote it by \( e(H) \). In general, \( e(H) \leq m(H) \).

We say that \( H \) has maximal embedding dimension if \( e(H) = m(H) \). The set \( G(H) = \mathbb{N} \setminus H \) is called the set of gaps of \( H \). Its cardinality is said to be the genus of \( H \) and we denote it by \( g(H) \).

A numerical semigroup \( H \) is symmetric if \( F(H) := \max \{ x \mid x \not\in H \} \) is odd and for every integer \( x \not\in H \), \( F(H) - x \in H \). We say that \( H \) is pseudo-symmetric if \( F(H) \) is even and for every \( x \not\in H \), either \( F(H) - x \in H \) or \( x = F(H)/2 \). It is known that a numerical semigroup is symmetric (resp. pseudo-symmetric) if and only if its semigroup ring is a Gorenstein (resp. Kunz) ring (see [BDF]). There are several characterizations of symmetric semigroups and pseudo-symmetric semigroups (see Lemma 2.2).

Let \( H = \langle a, b, c \rangle \) be a numerical semigroup generated by 3 elements and \( R = k[H] = k[t^a, t^b, t^c] \) be its semigroup ring over a field \( k \). We denote by \( m = (t^s \mid s \in H, s > 0) \) the unique graded maximal ideal of \( k[H] \). Let \( p = p(a, b, c) \) be the kernel of the homomorphism \( \varphi: S = k[X, Y, Z] \rightarrow R \) of \( k \)-algebras defined by \( \varphi(X) = t^a \), \( \varphi(Y) = t^b \), and \( \varphi(Z) = t^c \). Then it is known that if \( H \) is not symmetric, then the ideal \( p = p(a, b, c) = \text{Ker}(\varphi) \) is generated by the maximal minors of the matrix

\[
\begin{pmatrix}
X^\alpha & Y^\beta & Z^\gamma \\
Y^{\beta'} & Z^{\gamma'} & X^{\alpha'}
\end{pmatrix}
\]

(1)
for some positive integers \( \alpha, \beta, \gamma, \alpha', \beta', \gamma' \). The main goal of this paper is to prove Theorem 4.1, which characterizes pseudo-symmetric semigroups. Namely, \( H \) is pseudo-symmetric if and only if \( \alpha = \beta = \gamma = 1 \) or \( \alpha' = \beta' = \gamma' = 1 \).

Also, we introduce the notion of atomic semigroups and calculate \( g(H) \) when \( H = \langle a, b, c \rangle \) and \( a = 3, 4, 5 \).

2. PRELIMINARIES

In this section, we recall some definitions and basic facts about pseudo-symmetric numerical semigroups and type of numerical semigroups.

**Definition 2.1.** Let \( H \) be a numerical semigroup.

(1) We say that an integer \( x \) is a pseudo-Frobenius number if \( x \not\in H, x + s \in H \) for all \( s \in H, s \neq 0 \). We denote by \( \text{PF}(H) \) the set of pseudo-Frobenius numbers of \( H \).

Since \( H_m^1(k[H]) \) is generated by \( \{ t^m \mid m \in \mathbb{Z} \setminus H \} \) as \( k[H] \) module, \( \{ t^x \mid x \in \text{PF}(H) \} \) generates the socle of \( H_m^1(k[H]) \).

(2) The cardinality in \( \text{PF}(H) \) is called the type of \( H \), denoted by \( t(H) \). Since \( \text{PF}(H) \) corresponds to the socle of \( H_m^1(k[H]) \), \( t(H) = r(k[H]) \), the Cohen-Macaulay type of \( k[H] \).

(3) The \( a \)-invariant of the semigroup ring \( k[H] \) ([GW]) is defined to be

\[
a(k[H]) = \max \{ n \mid [H_m^1(k[H])]_n \neq 0 \}.
\]

Hence we have \( F(H) = a(k[H]) \).

It is easy to check that \( F(H) \in \text{PF}(H) \). From this we obtain that the numerical semigroup \( H \) is symmetric if and only if \( t(H) = 1 \). Also we note that

\[
g(H) = \dim(H_m^1(R)_{>0}) = \dim(k[\mathbb{Z} \setminus H]_{>0}).
\]

Next we will recall fundamental properties of pseudo-symmetric numerical semigroups.

**Lemma 2.2.** [FGH] Let \( H \) be a numerical semigroup. Then the following properties are equivalent.

1. \( H \) is pseudo symmetric.
2. \( F(H) \) is even and \( H \) is maximal among the set of all numerical semigroups with Frobenius number \( F(H) \).
3. \( \text{PF}(H) = \{ F(H), F(H)/2 \} \).

From Lemma 2.2 (3), if \( H \) is pseudo-symmetric, then \( t(H) = 2 \). But converse of this result is not true. For instance, if \( H = \langle 4, 5, 11 \rangle \), then \( \text{PF}(H) = \{ 6, 7 \} \), \( t(H) = 2 \) but it is not pseudo-symmetric since \( F(H) = 7 \) is odd.

**Theorem 2.3.** [FGH], [He] Let \( H \) be a numerical semigroup with \( e(H) = 3 \). Then the type of \( H \) does not exceed two.

3. NUMERICAL SEMIGROUPS GENERATED BY 3 ELEMENTS

We now let \( H = \langle a, b, c \rangle \) and assume \( H \) is not symmetric. Let \( R = k[H] = k[t^a, t^b, t^c] \cong k[X, Y, Z]/\mathfrak{p}(a, b, c) \) be its semigroup ring over a field \( k \). Then it is
known that the ideal $p = p(a, b, c)$ of $S = k[X, Y, Z]$ is generated by the maximal minors of the matrix

$$
\begin{pmatrix}
X^\alpha & Y^\beta & Z^\gamma \\
Y^\beta' & Z^\alpha' & X^\gamma' \\
\end{pmatrix},
$$

where $\alpha, \beta, \gamma, \alpha', \beta'$, and $\gamma'$ are positive integers.

Since $k[H]/(t^a) \cong k[Y, Z]/(Y^{\beta+\beta'}, Z^{\gamma+\gamma'})$, the defining ideal of $k[H]/(t^a)$ is generated by the maximal minors of the matrix

$$
\begin{pmatrix}
0 & Y^\beta & Z^\gamma \\
Y^\beta' & Z^\alpha' & 0 \\
\end{pmatrix}.
$$

Since $a = \dim_k k[H]/(t^a) = \dim_k k[Y, Z]/(Y^{\beta+\beta'}, Z^{\gamma+\gamma'})$, we get the equation $a = \beta\gamma + \beta'\gamma + \beta'\gamma'$. Similarly, we obtain that $b = \alpha\gamma + \alpha'\gamma' + \alpha'\gamma$ and $c = \alpha\beta + \alpha'\beta + \alpha'$. We put $l = Z^\gamma + Y^\beta m = X^\alpha + Y^\beta Z^\gamma$, and $n = Y^\beta + X^\alpha Z^\gamma$. There is the obvious relations

$$X^\alpha l + Y^\beta m + Z^\gamma n = Y^\beta l + Z^\gamma m + X^\alpha n = 0.$$

We put $p = \deg(X^\alpha Y^\beta) = \deg(Z^\gamma + Y^\beta)$, $q = \deg(X^\alpha + Y^\beta) = \deg(Y^\gamma + Z^\gamma)$, $r = \deg(Y^\beta + Z^\gamma) = \deg(X^\alpha + Z^\gamma)$, $s = \deg(X^\alpha) + \deg(Y^\gamma) + q = \deg(Y^\gamma) + r$, $t = \deg(Y^\beta) + p = \deg(Z^\gamma) + q = \deg(X^\alpha) + r$. Since $pd_S(R) = 2$, we get a free resolution of $R$

$$0 \to S(-s) \oplus S(-t) \to S(-p) \oplus S(-q) \oplus S(-r) \to S \to R \to 0.$$

Taking $\text{Hom}_S(\ast, K_S) = \text{Hom}_S(\ast, S(-a - b - c))$, we get

$$0 \to S(-x) \to S(p - x) \oplus S(q - x) \oplus S(r - x) \to S(s - x) \oplus S(t - x) \to K_R \to 0,$$

where $x = a + b + c$ and $K_R = \text{Ext}^2_S(R, S(-x))$. From this exact sequence, we have that $PF(H) = \{s - x, t - x\}$.

**Example 3.1.** Let $H = \langle 7, 12, 15 \rangle$ be a numerical semigroup and $R = k[H] \cong S/p$ be its semigroup ring over a field $k$. Then the ideal $p$ is generated by the maximal minors of the matrix

$$
\begin{pmatrix}
X^3 & Y^2 & Z^2 \\
Y & Z & X^3 \\
\end{pmatrix}.
$$

Hence we obtain that $F(H) = 32$, since $p = 45$, $q = 42$, $r = 36$, $s = 66$, $t = 57$ and $x = 34$.

Since ideal $p$ is generated by the maximal minors of the matrix

$$
\begin{pmatrix}
X^\alpha & Y^\beta & Z^\gamma \\
Y^\beta' & Z^\alpha' & X^\gamma' \\
\end{pmatrix},
$$

we obtain the following result.

**Lemma 3.2.** If $H = \langle a, b, c \rangle$ is not symmetric, then

1. $(\alpha + \alpha') a = \beta' b + \gamma c$ and $\alpha + \alpha' = \min\{n \mid an \in \langle b, c \rangle\},$
2. $(\beta + \beta') b = \alpha a + \gamma' c$ and $\beta + \beta' = \min\{n \mid bn \in \langle a, c \rangle\},$
3. $(\gamma + \gamma') c = \alpha' a + \beta b$ and $\gamma + \gamma' = \min\{n \mid cn \in \langle a, b \rangle\}.$

**Lemma 3.3.** If $H = \langle a, b, c \rangle$ is not symmetric, then $PF(H) = \{f, f'\}$, where

1. $f = \alpha a + (\gamma + \gamma') c - (a + b + c),$
2. $f' = \beta' b + (\gamma + \gamma') c - (a + b + c).$

**Proof.** Since $f = s - x$, $f' = t - x$, we obtain that $f = \deg(X^\alpha) + \deg(X^\alpha Y^\beta) - (a + b + c) = \alpha a + \alpha' a + \beta b - (a + b + c) = \alpha a + (\gamma + \gamma') c - (a + b + c)$ and $f' = \deg(Y^\beta) + \deg(X^\alpha Y^\beta) - (a + b + c) = \beta' b + (\gamma + \gamma') c - (a + b + c)$ by Lemma 3.2. \(\square\)
4. A CHARACTERIZATION OF PSEUDO-SYMMETRIC SEMIGROUPS

Now, we can prove our main theorem.

**Theorem 4.1.** $H = \langle a, b, c \rangle$ is pseudo symmetric if and only if

1. If $\beta' b > \alpha a$, then $\alpha = \beta = \gamma = 1$ and
2. If $\beta' b < \alpha a$, then $\alpha' = \beta' = \gamma' = 1$.

**Proof.** As is shown in Lemma 3.3, $PF(H) = \{ f, f' \}$ and we may assume $f < f'$. If $H$ is pseudo symmetric, then $2f = F(H) = f'$ by 2.2. We obtain that $(\beta' + 1)b = (2\alpha - 1)a + (\gamma + \gamma' - 1)c$. Since $2\alpha - 1 > 0$ and $\gamma + \gamma' - 1 > 0$, we get $\beta = 1$ and $(\beta' + 1)b = (2\alpha - 1)a + (\gamma + \gamma' - 1)c = \alpha a + \gamma c$ (by Lemma 3.2). Therefore $\alpha = \beta = \gamma = 1$.

Conversely, the condition 1) implies $f = \gamma' c - b$, $f' = f + \beta' b - a$. From Lemma 3.2, it follows that $\beta' b - a = \gamma' c - b = f$. Hence $f' = 2f$. This proves $H$ is pseudo symmetric. □

From Theorem 4.1, we obtain a characterization of pseudo-symmetric numerical semigroups with multiplicity three, four and five. Henceforth in this section, we assume $H = \langle a, b, c \rangle$ with $a < b < c$.

**Corollary 4.2.** Assume $a = 3$. Then $H$ is pseudo-symmetric if and only if $c = 2b - 3$.

**Proof.** Since $a = \beta \gamma + \beta' \gamma + \beta' \gamma' = 3$, we have that $\beta = \beta' = \gamma = \gamma' = 1$, $b = 2\alpha + \alpha'$ and $c = \alpha + 2\alpha'$. Hence $c = 2b - 3\alpha$. Since $b < c$ implies $\alpha < \alpha'$, we get $\beta' b > 3\alpha$. From Theorem 4.1, $H$ is pseudo-symmetric if and only if $\alpha = 1$, that is, $c = 2b - 3$. □

**Corollary 4.3.** Assume $a = 4$. Then $H$ is pseudo-symmetric if and only if $c$ is odd and $c = b + 2$.

**Proof.** Since $a = \beta \gamma + \beta' \gamma + \beta' \gamma' = 4$, among $\beta, \beta', \gamma, \gamma'$, either $\beta$ or $\gamma'$ is 2 and the rest is 1. But if $\gamma' = 2$, then $b > c$, contradicting our assertion. Hence we have $\beta' = \gamma = \gamma' = 1$ and $\beta = 2$ and then $b = 2\alpha + \alpha'$ and $c = 2\alpha + 3\alpha'$. Hence by 4.1 $\alpha' = 1$ and we get $c = b + 2$. □

**Corollary 4.4.** Assume $a = 5$. Then $H$ is pseudo-symmetric if and only if

1. $H = \langle 5, 2\alpha + 1, 3\alpha + 4 \rangle$ with $\alpha \geq 3, \alpha \neq 2 \pmod{5}$ or
2. $H = \langle 5, \alpha' + 2, 3\alpha' + 1 \rangle$ with $\alpha' \geq 4, \alpha' \neq 3 \pmod{5}$.

**Proof.** Since $a = 5 = \beta \gamma + \beta' \gamma + \beta' \gamma'$ and either $\alpha = \beta = \gamma = 1$ or $\alpha' = \beta' = \gamma' = 1$, we have to consider the following four cases.

- If $\beta = \beta' = \gamma = 1$ and $\gamma = 3$, then $b = 4\alpha + 3\alpha' > c = \alpha + 2\alpha'$, contradicting our assertion.
- If $\alpha' = \beta' = \gamma' = 1$ and $\gamma = 2$, then $b = 3\alpha + 1 > c = \alpha + 2$, contradicting our assertion.
- If $\alpha' = \beta' = \gamma = \gamma' = 1$ and $\beta = 3$, then $b = 2\alpha + 1$ and $c = 4\alpha + 3$.
- If $\alpha = \beta = \gamma = \gamma' = 1$ and $\beta' = 2$, then $b = \alpha' + 2$ and $c = 3\alpha' + 1$. □
5. Atomic Numerical Semigroups

Definition 5.1. A numerical semigroup $H$ is atomic if it is cannot be expressed as an intersection of two numerical semigroups with the same Frobenius number $F(H)$ properly containing $H$.

The concept of atomic numerical semigroups was introduced by J. C. Rosales [R2]. Our goal in this section is to study atomic numerical semigroups, especially when its embedding dimension is three. First we introduce some notation and basic properties for atomic numerical semigroups.

Definition 5.2. Let $H$ be a numerical semigroup. Then we put $SG(H) = \{ x \in PF(H) \mid 2x \in H \}$. We call elements of $SG(H)$ special gaps.

This concept is the key tool in this section. The following result is easy to show.

Lemma 5.3. [RG3] Let $H$ be a numerical semigroup and let $x \in G(H)$. Then $x \in SG(H)$ if and only if $H \cup \{x\}$ is a numerical semigroup.

An atomic numerical semigroup is characterized by the number of special gaps.

Lemma 5.4. [R2] A numerical semigroup $H$ is atomic if and only if $\#SG(H) \leq 2$.

The following result can be considered as the generalization of Definition 5.1.

Definition 5.5. Let $H$ be a numerical semigroup. We say $H$ is irreducible if it cannot be expressed as the intersection of two numerical semigroups properly containing $H$.

By definition, an irreducible numerical semigroup is clearly atomic. There is also characterization for irreducible numerical semigroups by the number of special gaps. The following result appears in [RGGJ]

Proposition 5.6. A numerical semigroups $H \neq \mathbb{N}$ is irreducible if and only if $\#SG(H) = 1$.

There is a nice characterization of irreducible semigroups.

Lemma 5.7. [FGH] Let $H$ be a numerical semigroup. The following conditions are equivalent.

1. $H$ is irreducible.
2. $H$ is maximal among the set of all numerical semigroups with Frobenius number $F(H)$.
3. $H$ is maximal among the set of all numerical semigroups that do not contain $F(H)$.

The symmetric and pseudo-symmetric numerical semigroups are characterized as irreducible semigroups.

Lemma 5.8. [FGH] Let $H$ be a numerical semigroup.

1. $H$ is symmetric if and only if $H$ is irreducible and $F(H)$ is odd.
2. $H$ is pseudo-symmetric if and only if $H$ is irreducible and $F(H)$ is even.
According to [R2] we call a numerical semigroup with \( \#SG(H) = 2 \) an **ANI-semigroup** (atomic not irreducible). The following result is useful to our study. There is a characterization for symmetric and pseudo-symmetric numerical semigroups in terms of the number of gaps.

**Proposition 5.9.** [FGH] Let \( H \) be a numerical semigroup. Then

1. \( H \) is symmetric if and only if \( g(H) = (F(H) + 1)/2 \).
2. \( H \) is pseudo-symmetric if and only if \( g(H) = (F(H) + 2)/2 \).

Every numerical semigroup with embedding dimension three is atomic.

**Lemma 5.10.** A numerical semigroup with embedding dimension three is atomic.

**Proof.** Since \( SG(H) \subset PF(H) \) is always true, it is clear by Theorem 2.3 and Lemma 5.4.

Hence if \( H \) is a ANI-semigroup with embedding dimension three, then \( SG(H) \) coincides with \( PF(H) \).

Now we consider the following questions.

**Question 5.11.** Let \( H \) be a ANI-semigroup with \( e(H) = 3 \).

1. How can we express \( g(H) \)? How far is \( H \) from irreducible semigroups?
2. In particular, if \( SG(H) = \{ x, F(H) \} \), when is \( H \cup \{ x \} \) irreducible?
3. When is \( H \cup \{ x \} \) generated by 3 elements.

By Lemma 5.7 and Proposition 5.9, \( H \cup \{ x \} \) is irreducible if and only if \( g(H) = (F(H) + 1)/2 + 1 \) (resp. \( g(H) = (F(H) + 2)/2 + 1 \) when \( F(H) \) is odd (resp. even). We study these questions when \( e(H) = 3 \) and \( m(H) = 3, 4 \).

First, we consider the case that \( m(H) = e(H) = 3 \). Namely, \( H = \langle a = 3, b, c \rangle \) with \( \gcd(3, b, c) = 1 \) and \( b < c \). Assume that \( p = p(3, b, c) \) is generated by the maximal minors of matrix

\[
\begin{pmatrix}
X^\alpha & Y^\beta & Z^\gamma \\
Y^{-\beta'} & Z^{-\gamma'} & X^{-\alpha'}
\end{pmatrix}.
\]

Since \( a = \beta \gamma + \beta' \gamma + \beta' \gamma' = 3 \), we have \( \beta = \beta' = \gamma = \gamma' = 1 \). Also, from Corollary 4.2, \( b = 2\alpha + \alpha' \) and \( c = \alpha + 2\alpha' \). Since \( b < c \), we must assume \( \alpha < \alpha' \). Then it is easy to see that

\[ f = b - 3 \quad \text{and} \quad f' = c - 3. \]

Since \( H \) has maximal embedding dimension, we have \( g(H) = \alpha + \alpha' - 1 \) by [RG3], Corollary 2.2. Hence we get the following.

**Proposition 5.12.** Let \( H = \langle 3, b, c \rangle \) with \( b < c \) and \( \alpha, \alpha' \) be as above with \( \alpha < \alpha' \) and \( \alpha' - \alpha \) is not a multiple of 3. Then

1. \( 2g(H) - F(H) = \alpha + 1 \).
2. \( H \) is pseudo-symmetric if and only if \( \alpha = 1 \) and \( \alpha' \geq 2 \).
3. \( H \) is not irreducible and \( H \cup \{ f \} \) is pseudo-symmetric (resp. symmetric) if and only if \( \alpha = 3 \) (resp. \( \alpha = 2 \)).

**Proof.** These results follow from calculation of \( g(H) \) and Proposition 5.9.

Here we give an example how an atomic numerical semigroup is related to an irreducible one.
Example 5.13. Let \( H = \langle 3, 28, 35 \rangle \) then \( k[H] \cong S/p \), where \( p = p(3, 28, 35) \) is generated by the maximal minors of the matrix

\[
\begin{pmatrix}
X^7 & Y & Z \\
Y & Z & X^{14}
\end{pmatrix}
\]

and we get \( SG(H) = \{25, 32\} \). If we repeat adding the smaller special gap to the semigroup, it will be pseudo-symmetric in three times. Indeed, \( H_1 = H \cup \{25\} = \langle 3, 25, 35 \rangle \) is not irreducible and \( SG(H_1) = \{22, 32\} \). Then \( H_2 = H_1 \cup \{22\} = \langle 3, 22, 35 \rangle \) is yet not irreducible and \( SG(H_2) = \{19, 32\} \). Then \( H_3 = H_2 \cup \{19\} = \langle 3, 19, 35 \rangle \) is finally irreducible.

Next we study the case \( e(H) = 3 \) and \( m(H) = 4 \).

Let \( H = \langle 4, b, c \rangle \) with \( b < c \). In the same manner as in the case \( m(H) = 3 \), we get \( \beta = 2, \beta' = \gamma = \gamma' = 1 \). Also, we have

\[
b = 2\alpha + \alpha', \quad c = 2\alpha + 3\alpha'.
\]

Hence \( \alpha' \) must be odd. In the same manner, we compute

\[
f = 4\alpha + 2\alpha' - 4, \quad f' = c - 4 = 2\alpha + 3\alpha' - 4.
\]

Note that \( F(H) = \max\{f, f'\} \) and \( f < f' \iff 2\alpha < \alpha' \). Since the Apery set of \( H \) of \( H \) in 4 is \( \{b, 2b, c\} \) (Apery set in 4 shows the least integer in \( H \) from each residue class modulo 4), we can compute \( g(H) \).

Proposition 5.14. Let \( H = \langle 4, b, c \rangle \) be a ANI-numerical semigroup and \( PF(H) = \{f, f'\} \). Then

1. \( g(H) = 2\alpha + 3(\alpha' - 1)/2 \).
2. (a) If \( F(H) = f' \), then \( H \cup \{f\} \) is symmetric if and only if \( \alpha = 1 \) and \( \alpha' \geq 3 \).
   (b) If \( F(H) = f \), then \( H \cup \{f'\} \) is pseudo-symmetric if and only if \( \alpha \geq 2 \) and \( \alpha' = 3 \).

Proof. If \( \alpha' > 2\alpha \), then \( F(H) = c - 4 \) and \( 2g(H) - F(H) = 2\alpha + 1 \). Hence in this case, \( H \cup \{f'\} \) is symmetric if and only if \( \alpha = 1 \). In this case, \( H \cup \{f'\} = \langle 4, b\alpha' + 2, 2\alpha' \rangle \).

Finally, we will give an answer to Question 5.11, (3). We use notation in Lemma 3.3. We always assume \( H = \langle a, b, c \rangle \) with \( a < b < c \) and also that \( H \) is ANI. Note that \( H \cup \{x\} = \langle a, b, c, f \rangle \) (resp. \( \langle a, b, c, f' \rangle \)) if \( f < f' \) (resp. \( f' < f \)).

Now, since \( f = (\alpha - 1)a + (\gamma + \gamma' - 1)c - b = (\alpha + \alpha' - 1)a + (\beta - 1)b - c \), \( e(H \cup \{x\}) = 3 \) if and only if either \( \alpha = \gamma = \gamma' = 1 \) and \( c = f + b \) or \( \alpha = \alpha' = \beta = 1 \) and \( a = c + f \). But since \( a < b < c \), the latter case does not occur and we have the following result.

Proposition 5.15. Let \( H = \langle a, b, c \rangle \) with \( a < b < c \) and assume that \( H \) is an ANI semigroup with \( PF(H) = \{f, f'\} \). Then putting \( x = \min\{f, f'\} \), \( e(H \cup \{x\}) = 3 \) if and only if the following conditions hold.

1. If \( f < f' \), then \( \alpha = \gamma = \gamma' = 1 \) and \( c = f + b \). In this case, \( p(a, b, f) = p(a, b, c - b) \) is generated by the maximal minors of the matrix

\[
\begin{pmatrix}
X & Y^\beta - 2 & Z \\
Y^\beta + 1 & Z & X^{\alpha'}
\end{pmatrix}
\]
(2) If \( f' < f \), then \( \beta' = \gamma = \gamma' = 1 \) and \( c = f + a \). In this case, \( p(a, b, f') = p(a, b, c-a) \) is generated by the maximal minors of the matrix
\[
\begin{pmatrix}
X^{\alpha+1} & Y^\beta & Z \\
Y & Z & X^{\alpha'-2}
\end{pmatrix}
\]

(3) If we repeat the process as in (1) (resp. (2)) \( k \) times until we have \( \beta - 2k = 0 \) or \( 1 \) (resp. \( \alpha' - 2k = 1 \) or \( 0 \)), we get an irreducible semigroup.

Acknowledgment. The third author was partially supported by Grant-in-Aid for Scientific Research 20540050 and Individual Research Expense of College of Humanity and Sciences, Nihon University.

REFERENCES


