On Rotation Number of Dynamical Systems

Makoto MORI

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1 Introduction

We will consider a piecewise monotone function $F_0$ from $\mathbb{R}$ into itself such that

1. for any $x_0 \in \mathbb{R}$, both limits $\lim_{x \uparrow x_0} F_0(x)$ and $\lim_{x \downarrow x_0} F_0(x)$ exist, and one of the above two limits coincide with $F(x_0)$,

2. there exists an integer $m$ such that for any $x_0 \in \mathbb{R}$

$$ \lim_{x \uparrow x_0} F_0(x + 1) = \lim_{x \downarrow x_0} F_0(x) + m, $$

3. Near the neighborhood of $0$ in $x > 0$, one of the followings holds:

(a) if $F$ is monotone increasing, then $0 \leq \lim_{x \downarrow 0} F_0(x) < 1$,

(b) if $F$ is monotone decreasing, then $0 < \lim_{x \downarrow 0} F_0(x) \leq 1$.

Then we can consider a transformation $F : [0, 1] \to [0, 1]$ by

$$ F(x) = F_0(x) \pmod{1}. $$

**Definition 1** For a dynamical system $([0, 1], \mu, F)$, let

$$ \rho(x) = \lim_{n \to \infty} \frac{F_0^n(x)}{n}. $$

If there exists a constant $\rho$ such that $\rho(x) = \rho$ a.e. $\mu$, then we call $\rho$ a rotation number of $F$ with respect to the measure $\mu$.

**Remark 1** The usual definition for an orientation preserving homeomorphism, we define the rotation number by $\rho(x) \pmod{1}$. But in this article, we will consider general transformations, so we assume $0 \leq \lim_{x \downarrow 0} F_0(x) \leq 1$.

When $F$ is an orientation preserving homeomorphism, we know the following Theorem 1 and Theorem 2 (see for example [2]).
Theorem 1 The limit
\[ \rho = \lim_{n \to \infty} \frac{F^n_0(x)}{n} \]
exists and does not depend on \( x \).

Theorem 2 The rotation number \( \rho \) is rational, if and only if \( F^n \) has a fixed point for some \( n \).

The main theorem which we will prove for general piecewise monotonic transformations is the following.

Theorem 3 Assume that the dynamical system \(([0, 1], \mu, F)\) is mixing. If \( r_K \) is \( C^1 \) in \( K \) in a neighborhood of \( K = 1 \), then the rotation number exists with respect to the invariant measure and
\[ \rho = \lim_{K \downarrow 1} \log_K \frac{r_1}{r_K}, \]
where \( r_K \) is the minimal zero of \( \det(I - \Phi^K(z)) \), and the definition of \( K \)-Fredholm matrix \( \Phi^K(z) \) will be given later.

2 Symbolic Dynamics

As we assume that \( F_0 \) is piecewise monotone, \( F \) is also piecewise monotone. Thus there exists a finite set \( \mathcal{A} \) such that an subinterval \( \langle a \rangle \) \((a \in \mathcal{A})\) corresponds and

1. \( \{\langle a \rangle\}_{a \in \mathcal{A}} \) is a partition of \([0, 1]\),
2. \( F \) restricted to \( \langle a \rangle \) is monotone and continuous.

We can consider the natural order on \( \mathcal{A} \) from the order on the interval.

We define an index of \( a \in \mathcal{A} \) by
\[ \text{id}_a = \lfloor F_0(x) \rfloor, \quad x \in \langle a \rangle, \]
and
\[ \text{id}(x) = \sum_{a \in \mathcal{A}} \text{id}_a 1_{\langle a \rangle}(x). \]

Remark that there exists \( M > 0 \) such that \( \max_{a \in \mathcal{A}} |\text{id}_a| \leq M \), because \( \mathcal{A} \) is a finite set.

Lemma 1
\[ F_0^n(x) - 1 < \sum_{k=0}^{n-1} \text{id}(F^k(x)) \leq F_0^n(x). \]
\textbf{Proof.} From the definition,

\[ [F_0(x)] - [x] = [F_0(x)] = \text{id}(x). \]

In general,

\[ [F_0^n(x)] - [F_0^{n-1}(x)] = \text{id}(F_0^{n-1}(x)). \]

Therefore

\[ [F_0^n(x)] = \sum_{k=0}^{n-1} \text{id}(F_0^k(x)). \]

This proves the lemma.

Thus instead of counting \( \frac{F_0^n(x)}{n} \), we need to calculate the mean

\[ \frac{1}{n} \sum_{k=0}^{n-1} \text{id}(F_0^k(x)), \]

whose limit exists almost everywhere if the dynamical system \([0,1], \mu, F\) is ergodic. Here we denote the invariant probability measure on the unit interval by \( \mu \).

Let us define for a word \( w = a_1 \cdots a_n \ (a_i \in A) \)

1. \(|w| = n, \)
2. \( \langle w \rangle = \bigcap_{i=1}^n F^{-i+1}(\langle a_i \rangle), \)
3. \( w[i] = a_i, \)
4. \( \eta^K_w = \prod_{i=1}^n K^{a_i}, \quad (K > 0). \)

For \( x \in [0,1] \), we define an expansion \( a_1^x a_2^x \cdots \) of \( x \) by

\[ F^i(x) \in \langle a_{i+1}^x \rangle, \quad (i \geq 0). \]

For a word \( w \) and a point \( x \), we define \( wx \) as an infinite series of symbols \( a_1 \cdots a_n a_1^x a_2^x \cdots \). If there exists a point \( y \in [0,1] \) such that its expansion coincides with \( wx \), then we call \( wx \) exists and identify with \( y \). We can identify the transformation \( F \) on \([0,1]\) with the shift \( \theta \) on a closed subset of \( A^\mathbb{N} \).

When we want to study ergodic properties of a dynamical system, the spectra of the Perron–Frobenius operator associated with the dynamical system is a good tool \([1, 3, 4, 5]\). To study it, we consider a generating function

\[ [s^0]_G(z) = \sum_{n=0}^\infty z^n \sum_{|w|=n} \int 1_f(wx) \eta^0_w g(x) \, dx, \]

where

\[ \eta^0_w = |F^m(x)|^{-1}, \quad x \in \langle w \rangle. \]
Then we can prove the spectra of the Perron–Frobenius operator is the reciprocal of the zeros of the singularities of the above generating function ([6, 7]), and we can determine the density of the invariant measure \( \mu \) by

\[
\lim_{z \to 1} (1 - z)[s^0]_g(z) = |J| \int g \, d\mu.
\]

Now, instead of the slope \( \eta_0^w \), we use index defined above. For an interval \( J \) and \( g \in L^\infty \), we define a generating function by

\[
[s^K]_g(z) = \sum_{n=0}^{\infty} z^n \sum_{|w|=n} \int_{J} 1_{J}(wx) \eta^K_w g(x) \, dx.
\]

When \( K \neq 1 \), \([s^K]_g(z)\) is used to estimate the growth rate of the index of \( wx \), and \([s^1]_g(z)\) is used to estimate the growth rate of the number of words when the length goes to infinity.

To study a general transformation, we will introduce signed symbolic dynamics. For an interval \( J \), we express \( J^+ = \lim_{x \in J, x \to \sup J} a_1^x a_2^x \cdots \) and \( J^- = \lim_{x \in J, x \to \inf J} a_1^x a_2^x \cdots \).

We will express \( \langle a \rangle^+ \) and \( \langle a \rangle^- \) by \( a^+ \) and \( a^- \), and denote

\[
\tilde{A} = \{a^\sigma\}_{a \in A, \sigma = +, -}.
\]

We consider a dynamical system on the set of infinite sequences of symbols with sign and call it signed symbolic dynamics. For an infinite sequence of symbols \( \alpha = a_1 a_2 \cdots \), we consider \( \alpha^\sigma \) with sign \( \sigma \in \{+, -\} \). We will express \( \tilde{\alpha} \) either \( \alpha^+ \) or \( \alpha^- \).

1. \( \alpha <^\sigma \beta = \begin{cases} \alpha < \beta, & \sigma = +, \\ \alpha > \beta, & \sigma = -. \end{cases} \quad (\alpha, \beta \in A^N), \]

2. \( \tilde{\alpha}_i = a_i, \)

3. \( \delta[L] = \begin{cases} +1 & \text{if } L \text{ is true}, \\ 0 & \text{otherwise}, \end{cases} \)

4. \( \epsilon(\theta \alpha^\sigma) = \epsilon(\alpha^\sigma) = \sigma, \)

5. \( \sigma(\alpha^\sigma, \beta) = \begin{cases} +\frac{1}{2} & \alpha^\sigma >^\sigma \beta, \\ -\frac{1}{2} & \alpha^\sigma <^\sigma \beta. \end{cases} \)

Here, we identify \( x \in [0, 1] \) with its expansion \( a_1^x a_2^x \cdots \).
Now, we define a generating function corresponding to signed symbolic dynamics by
\[
[s^K]_g(\tilde{\alpha})(z) = \epsilon(\tilde{\alpha}) \left( \int \sigma(\tilde{\alpha}, x)g(x)
\right. \\
+ \sum_{n=1}^{\infty} \sum_{\|w\|=n} \int \eta^K_w \delta[w[1] = \alpha_1, \exists \theta wx] \sigma(\tilde{\alpha}, wx)g(x)
\bigg). 
\]

Then we get the following lemma (cf.\cite{6, 7}).

**Lemma 2** For an interval \( J \subset \langle a \rangle \) \((a \in \mathcal{A})\), then
\[
[s^K]_g(z) = [s^K]_g^+(z) + [s^K]_g^-(z).
\]

To calculate \( [s^K]_g^+(z) \), we need to trace the orbits of both endpoints of \( J \). But \( [s^K]_g^-(z) \) depend only on the orbit of the endpoint \( J^- \). Thus from the above Lemma 2, we can calculate \( [s^K]_g(z) \) by tracing the orbits of the endpoints of \( J^- \) separately.

We will introduce renewal equations of our generating functions. For an infinite sequence of symbols with sign \( \tilde{\alpha} \) and \( \tilde{b} \in \tilde{A} \), we define

1. \([\chi^K]_{\tilde{\alpha}}(z, x) = \epsilon(\tilde{\alpha}) \sum_{n=0}^{\infty} z^n \eta^K_{\tilde{\alpha}_1 \cdots \tilde{\alpha}_n} \sigma(\theta^n \tilde{\alpha}, x),
\]

2. \(\phi(\tilde{\alpha}, \tilde{b}) = \begin{cases} 
+\frac{1}{2} & \tilde{b} \leq \langle \tilde{\alpha}_2 \rangle^- \\
-\frac{1}{2} & \tilde{b} > \langle \tilde{\alpha}_2 \rangle^- 
\end{cases} \)

3. \(K\)–Fredholm matrix is defined by \(\Phi^K(z)_{\tilde{\alpha}, \tilde{b}} = \epsilon(\tilde{\alpha}) \sum_{n=1}^{\infty} z^n \operatorname{sgn}(\tilde{\alpha}_1 \cdots \tilde{\alpha}_n) \eta^K_{\tilde{\alpha}_1 \cdots \tilde{\alpha}_n} \phi(\theta^n \tilde{\alpha}, \tilde{b}).\)

Now we define vectors
\[
[s^K]_g(z) = (\mathcal{K}^{\tilde{\alpha}})^{\tilde{\alpha} \in \tilde{A}}
\]
\[
[\chi^K]_g(z) = \left( \int [\chi^K]_{\tilde{\alpha}}(z, x)g(x)
\right. \\
\bigg)_{\tilde{\alpha} \in \tilde{A}}.
\]

Then we get

**Lemma 3**
\[
[s^K]_g(z) = (I - \Phi^K(z))^{-1}[\chi^K]_g(z).
\]

**Proof.** Divide
\[
[s^K]_g(z) = \epsilon(\tilde{\alpha}) \left( \int \sigma(\tilde{\alpha}, x)g(x)
\right. \\
+ \sum_{n=1}^{\infty} \sum_{\|w\|=n} \int \eta^K_w \delta[w[1] = \alpha_1, \exists \theta wx] \sigma(\tilde{\alpha}, wx)g(x)
\bigg). 
\]
Then taking \( n \) to \( n - 1 \), and \( w \) to \( \tilde{\alpha}_1 w \), the latter part equals

\[
\epsilon(\tilde{\alpha}) z \eta^K_{\tilde{\alpha}_1} \sum_{n=0}^{\infty} z^n \sum_{|w|=n} \int \eta^K_w \sigma(\theta \tilde{\alpha}, wx) g(x) \, dx.
\]

Then the above formula equals

\[
\epsilon(\tilde{\alpha}) z \eta^K_{\tilde{\alpha}_1} \sum_{n=0}^{\infty} z^n \sum_{|w|=n} \int \eta^K_w \text{sgn} \tilde{\alpha}_1 \left[ \left( \sum_{b \in A, b < \tilde{\alpha}_2} \frac{1}{2} - \sum_{b \in A, b > \tilde{\alpha}_2} \frac{1}{2} \right) \delta_{(b)}(wx) + \frac{1}{2} \delta_{(\tilde{\alpha}_2 - \theta \tilde{\alpha})}(wx) - \frac{1}{2} \delta_{(\tilde{\alpha}_2, \theta \tilde{\alpha})}(wx) \right] g(x) \, dx
\]

\[
= \frac{1}{2} \epsilon(\tilde{\alpha}) z \eta^K_{\tilde{\alpha}_1} \text{sgn} \tilde{\alpha}_1 \times \left( \sum_{b \in A, b < \tilde{\alpha}_2} [s^K_g]_{\tilde{b}}(z) - \sum_{b \in A, b > \tilde{\alpha}_2} [s^K_g]_{\tilde{b}}(z) + [s^K_g]_{(\tilde{\alpha}_2 - \theta \tilde{\alpha})}(z) - [s^K_g]_{\theta \tilde{\alpha}, (\tilde{\alpha}_2 + \theta \tilde{\alpha})}(z) \right).
\]

Then using Lemma 2, we can divide \([s^K] \) into signed symbolic dynamics and we get

\[
= z \eta^K_{\tilde{\alpha}_1} \sum_{b \in A} \epsilon(\tilde{\alpha}) \text{sgn} \tilde{\alpha}_1 \phi(\theta \tilde{\alpha}, \tilde{b}) [s^K_g]_{\tilde{b}}(z) + z \text{sgn} \tilde{\alpha}_1 \eta^K_{\tilde{\alpha}_1} [s^K_g]_{\theta \tilde{\alpha}, (\tilde{\alpha}_2 + \theta \tilde{\alpha})}(z).
\]

Then continue this procedure, we get the proof (see detail [6]).

Thus the radius of convergence \( r_K \) of the above generating function is the minimal zero of \( \det(I - \Phi^K(z)) \) in modulus. Especially, for an expanding map, the minimal zero of \( \det(I - \Phi^0(z)) \) equals \( r_0 = 1 \), and if this zero is simple, then the dynamical system is ergodic. Let us denote the absolute value of the second smallest zero of \( \det(I - \Phi^0(z)) \) in modulus by \( \delta^{-1} \). Then \( \delta \) determines the decay rate of correlation:

**Lemma 4** For \( f \in L^\infty \) and \( g \in BV \),

\[
\left| \int g(x) f(F^n(x)) \, dx - \int g(\cdot) \times \int f \, d\mu \right| < C_0 \delta^n \| g \|_{BV} \| f \|_{\infty},
\]

with some constant \( C_0 \). Here \( \| \cdot \|_{\infty} \) is the \( L^\infty \) norm and \( \| \cdot \|_{BV} \) is the norm which is the sum of \( L^1 \) norm and the total variation.

For the proof, see [6, 7].

Hereafter assume that the dynamical system is mixing.

**Lemma 5**

\[
| \int \sum_{k=0}^{n-1} (\text{id}(F^k(x)) - \rho) g(x) \, dx | \leq 2C_0 M \| g \|_{BV} \frac{1}{1-\delta}.
\]
Proof. Because the dynamical system is ergodic,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \text{id}(F^k(x)) = \rho
\]

holds almost everywhere. Hence

\[
\int (\text{id}(x) - \rho) \, d\mu = 0.
\]

Thus we get

\[
\left| \int \sum_{k=0}^{n-1} (\text{id}(F^k(x)) - \rho) \, g(x) \, dx \right| \leq \sum_{k=0}^{n-1} \left| \int (\text{id}(F^k(x)) - \rho) \, g(x) \, dx \right|
\]

\[
\leq C_0 \sum_{k=0}^{n-1} \delta^k ||g||_{BV} ||\text{id} - \rho||_\infty
\]

\[
\leq 2C_0 M ||g||_{BV} \frac{1}{1 - \delta}.
\]

Now we take \( g \equiv 1 \).

Lemma 6 For any \( \varepsilon > 0 \) there exists \( n_0 \) such that for any \( n \geq n_0 \)

\[
\sum_{k=0}^{n-1} \int (\text{id}(F^k(x)) - \rho - \varepsilon) \, dx \leq -\frac{n}{2} \varepsilon,
\]

\[
\sum_{k=0}^{n-1} \int (\text{id}(F^k(x)) - \rho + \varepsilon) \, dx \geq +\frac{n}{2} \varepsilon.
\]

Proof.

\[
\sum_{k=0}^{n-1} \int (\text{id}(F^k(x)) - \rho - \varepsilon) \, dx
\]

\[
= \sum_{k=0}^{n-1} \int (\text{id}(F^k(x)) - \rho) \, dx - n\varepsilon \int dx
\]

\[
\leq 2C_0 M \frac{1}{1 - \delta} - n\varepsilon.
\]

Thus taking \( n \) sufficiently large, we get the proof.
Lemma 7  Let
\[ J_n^+ = \{ x : \sum_{k=0}^{n-1} (\text{id}(F^k(x)) - \rho - \varepsilon) < 0 \}, \]
\[ J_n^- = \{ x : \sum_{k=0}^{n-1} (\text{id}(F^k(x)) - \rho + \varepsilon) > 0 \}. \]

Then we get for \( n \geq n_0 \)
\[ |J_n^-|, |J_n^+| > \frac{\varepsilon}{2(2M+\varepsilon)}. \]

Proof. We will prove the first assertion. Note that \(|\rho| \leq M\) because \(|\text{id}(x)| \leq M\).
\[
\sum_{k=0}^{n-1} \int (\text{id}(F^k(x)) - \rho - \varepsilon) \, dx \geq \int_{J_n^+} \sum_{k=0}^{n-1} (\text{id}(F^k(x)) - \rho - \varepsilon) \, dx \\
\geq n(-2M - \varepsilon) \int_{J_n^+} \, dx \\
\geq -(2M + \varepsilon)n|J_n^-|.
\]

Then by Lemma 6, we get for \( n \geq n_0 \)
\[ -\frac{n}{2} \varepsilon \geq \sum_{k=0}^{n-1} \int (\text{id}(F^k(x)) - \rho - \varepsilon) \, dx \geq -(2M + \varepsilon)n|J_n^-|. \]

This shows
\[ |J_n^-| \geq \frac{\varepsilon}{2(2M+\varepsilon)}. \]

Another inequality can be proved in a same way.

Lemma 8  Assume that \( h(x) \geq \Delta > 0 \). Let
\[ a_n = \int \sum_{|w| = n} K^{\sum_{k=0}^{n-1} \text{id}(F^k(x))} h(x) \, dx. \]

Then the radius of convergence of \( \sum_{n=0}^{\infty} a_n z^n \) is less than or equal to \( K^{-\rho}r_1 \).

Proof. From the definition, we get
\[ \limsup_{n \to \infty} (\# \{|w| = n\})^{1/n} = \frac{1}{r_1}. \]
Assume now $K > 1$. Take any $\varepsilon > 0$. Then by Lemma 7

\[
\int \sum_{|w| = n} K^{\sum_{k=0}^{n-1} \text{id}(F^k(x))} h(x) \, dx
\]

\[
= K^{(\rho - \varepsilon)n} \sum_{|w| = n} \int K^{\sum_{k=0}^{n-1} \text{id}(F^k(x)) - \rho + \varepsilon} h(x) \, dx
\]

\[
\geq K^{(\rho - \varepsilon)n} \sum_{|w| = n} \int_{J_+^n} h \, dx
\]

\[
\geq K^{(\rho - \varepsilon)n} \sum_{|w| = n} |J_+^n| \Delta
\]

\[
\geq K^{(\rho - \varepsilon)n} \frac{\varepsilon}{2(2M + \varepsilon)} \Delta \#\{|w| = n\}.
\]

Hence

\[
\frac{1}{r_K} = \limsup_{n \to \infty} \left( \int \sum_{|w| = n} 2^{- \sum_{k=0}^{n-1} \text{id}(F^k(x))} h(x) \, dx \right)^{1/n} \geq K^{\rho - \varepsilon} \frac{1}{r_1}.
\]

We can take $\varepsilon > 0$ arbitrary small, thus we get the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ is less than or equal to $K^{-\rho} r_1$.

When $1 > K > 0$,

\[
\int \sum_{|w| = n} K^{\sum_{k=0}^{n-1} \text{id}(F^k(x))} h(x) \, dx
\]

\[
\geq K^{(\rho + \varepsilon)n} \sum_{|w| = n} \int_{J_-^n} K^{\sum_{k=0}^{n-1} \text{id}(F^k(x)) - \rho - \varepsilon} h(x) \, dx
\]

\[
\geq K^{(\rho + \varepsilon)n} \sum_{|w| = n} \int_{J_-^n} h \, dx
\]

\[
\geq K^{(\rho + \varepsilon)n} \frac{\varepsilon}{2(2M + \varepsilon)} \Delta \#\{|w| = n\}.
\]

From this, we get the proof in a similar way.

**Lemma 9** For $K > 1$, we get

\[
\frac{r_1/K}{r_1} \leq K^\rho \leq \frac{r_1}{r_K}.
\]

Moreover,

\[
r_K \times \frac{1}{r_1} = (r_1)^2,
\]
holds, then the rotation number satisfies
\[ \rho = \log_K \left( \frac{r_1}{r_K} \right) = \log_K \left( \frac{r_1/K}{r_1} \right). \]

**Proof.** By Lemma 8, we get
\[ r_1/K \leq K^\rho r_1 \quad \text{and} \quad r_K \leq K^{-\rho} r_1. \]

Thus
\[ \frac{r_1/K}{r_1} \leq K^\rho \leq \frac{r_1}{r_K}. \]

When \( r_K \times r_1/K = (r_1)^2 \), then we get
\[ \frac{r_1/K}{r_1} = K^\rho = \frac{r_1}{r_K}. \]

This proves the lemma.

**Proof of Theorem 3.** Take the Taylor expansion,
\[ r_K = r_1 + O(K - 1). \]

Then
\[ \left( \frac{r_1}{r_K} - \frac{r_1/K}{r_1} \right) \times r_K \times r_1 = r_1^2 - r_K \times r_1/K \]
\[ = r_1^2 - (r_1 + O(K - 1)) \times (r_1 + O(\frac{1}{K} - 1)) \]
\[ = O(K - 1). \]

This shows
\[ \lim_{K \downarrow 1} \frac{r_1}{r_K} = \lim_{K \downarrow 1} \frac{r_1/K}{r_1}. \]

Hence by Lemma 9, we get the proof.

**Example 1** Let \( F \) be a homeomorphism. Then, by Theorem 1, we need not the assumption that the dynamical system is ergodic and \( r_1 = 1 \).

Let \( \mathcal{A} = \{a, b\} \), that is, \( \inf \langle a \rangle = 0 \), \( \sup \langle a \rangle = 1 \), \( \langle a \rangle \cup \langle b \rangle = [0, 1] \) and \( \langle a \rangle \cap \langle b \rangle = \emptyset \). Moreover, we assume \( F(a^-) = F(b^+) \), \( F(a^+) = 1 \) and \( F(b^-) = 0 \).

When \( \theta a^- = \theta b^+ = s_2 s_3 \ldots \), define
\[ a(z) = \sum_{n=2}^{\infty} z^n K^\# \{1 \leq i \leq n : s_i = b \} - 1 \]
\[ b(z) = \sum_{n=2}^{\infty} z^n K^\# \{1 \leq i \leq n : s_i = b \} - 1 \times (2\delta[s_n = a] - 1) \]
Then

$$\Phi^K(z) = \begin{pmatrix}
  -z/2 - a(z) & -z/2 - b(z) & z/2 - b(z) & z/2 + a(z) \\
  z/2 & z/2 & z/2 & z/2 \\
  z + a(z) & z + b(z) & z + b(z) & -z - a(z)
\end{pmatrix}.$$ 

Remark 2 When $F(\tilde{\alpha}) = \tilde{b}$ ($\tilde{b} \in \tilde{A}$), then we can simplify the Fredholm matrix, because

$$[s^K]_{\tilde{g}}(z) = \epsilon(\tilde{\alpha}) \int \sigma(\tilde{\alpha}, x) g(x) \, dx + \epsilon(\tilde{\alpha}) z \eta^K_{\tilde{\alpha}1} \text{sgn} \tilde{\alpha}_1 \sigma(\theta \tilde{\alpha}, \tilde{b}) [s^K]_{\tilde{g}}(z),$$

that is, $\Phi^K(z)_{\tilde{\alpha}, \tilde{b}} = \epsilon(\tilde{\alpha}) z \eta^K_{\tilde{\alpha}1} \text{sgn} \tilde{\alpha}_1 \sigma(\theta \tilde{\alpha}, \tilde{b})$.

Example 2 As in Example 1, let $F$ be a homeomorohism, and assume moreover $F(a^-) = F(b^+) = a^+$. Then

$$\Phi^K(z)_{\tilde{\alpha}, \tilde{b}} = \epsilon(\tilde{\alpha}) z \eta^K_{\tilde{\alpha}1} \text{sgn} \tilde{\alpha}_1 \sigma(\theta \tilde{\alpha}, \tilde{b}).$$

Thus the Fredholm determinant $\det (I - \Phi^K(z)) = 1 - Kz^2$. Hence, $r_K = \frac{1}{\sqrt{K}}$ and $r_1 = 1$. Thus $r_K \times r_1/K = 1 = (r_1)^2$ holds. Therefore, the rotation number equals

$$\rho = \log_K \sqrt{K} = \frac{1}{2}.$$
Figure 2: an example of non–homeomorphic transformation

\[ \Phi^K(z) = \begin{pmatrix} z/2 & z/2 & z/2 & z/2 \\ z/2 & z/2 & z/2 & z/2 \\ Kz/2 & Kz/2 & Kz/2 & Kz/2 \\ Kz/2 & Kz/2 & -Kz/2 & -Kz/2 \end{pmatrix}. \]

Thus \( r_K = \frac{-1 + \sqrt{1 + 4K}}{2K} \) and \( r_1 = \frac{-1 + \sqrt{5}}{2} \). When \( K = 2 \), \( r_2 = \frac{1}{2} \) and \( r_{1/2} = \sqrt{3} - 1 \). If the dynamical system is mixing, then the rotation number satisfies

\[ \frac{2(\sqrt{3} - 1)}{-1 + \sqrt{5}} \leq \rho \leq \sqrt{5} - 1. \]

Hence the rotation number is between 0.244 and 0.306. Now we apply Theorem 3.

\[ \log_K \frac{r_1}{r_K} = \frac{\log(-1 + \sqrt{5})/2 - \log(-1 + \sqrt{1 + 4K})/2K}{\log K} \]

\[ = \left( \frac{5 - \sqrt{5}}{10} \right) + O(K - 1) \]

Therefore, if the dynamical system is mixing, the rotation number equals \( \frac{5 - \sqrt{5}}{10} (= 0.276) \).

References


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