Minimizing movement of free discontinuity set

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0. Introduction

In 1988, DeGiorgi and Ambrosio have introduced in the article [DGA] the notion of $SBV$-functions, that is, the special functions of bounded variation, first motivated by the image segmentation problem proposed by Mumford and Shah [MS]. $SBV$-function is, roughly speaking, $W^{1,1}$-Sobolev function with the essential discontinuity in the interior of the domain of definition. Due to Ambrosio [Amb1-3], established are the compactness property of $SBV$-function space and lower semicontinuities for mutually disjointed components of the total variation measure of $SBV$-function.

In the paper [DCL], a variational functional defined for $SBV$-functions has been pursued, and on the basis of the investigation the image segmentation problem turned out to be solvable in general dimension. In particular, Tamanini, Congedo and Massari have studied the problem for the case in which the admissible $SBV$-functions are locally constant ([CT1,2], [MT1,2]). This problem, called minimal boundary problem, has been originated in [MS, Chapter5].

In this paper, we shall concern ourselves with the time-evolution of the problem related to one by Tamanini et al as stated above: Let $\Omega$ be a bounded domain in $n$-dimensional Euclidean space $\mathbb{R}^n$, $n \geq 2$. Suppose $\mathbf{T}$ to be a subset of $\mathbb{R}$ such that the cardinality of the set $\mathbf{T} \cap [-L,L]$ is finite for any positive number $L$. Set $BV(\Omega, \mathbf{T}) = \{u \in BV(\Omega) | u(x) \in \mathbf{T} \text{ for } \mathcal{L}^n\text{-almost all } x \in \Omega\}$, where $BV(\Omega)$ is the space of real valued functions of bounded variation defined on $\Omega$ (to be precise, refer to [Gi], [M-M]). Henceforth, we shall denote by $\mathcal{L}^n$ the $n$-dimensional Lebesgue measure in $\mathbb{R}^n$. We define

$$V(\Omega, \mathbf{T}) := \left\{ v \in BV(\Omega, \mathbf{T}) \left| \int_\Omega |v|^2 d\mathcal{L}^n + \mathcal{H}^{n-1}(S_v) < \infty \right. \right\},$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$ (see for instance [LS]). $S_v$, called the discontinuity set of $v$, is the set of all the points on which $v$ is essentially discontinuous. It is well known that the discontinuity set of $BV$-function is countably $(n-1)$-rectifiable set (see [Amb2], and refer to [Fe], [LS] for the terminology 'rectifiable set').

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Suppose that \( g \) and \( u_0 \) are bounded \( \mathcal{L}^n \)-measurable functions defined on \( \Omega \) and assume that \( u_0 \in V(\Omega, T) \). The main ingredient treated in the present paper, is a sequence of functions \( \{u_{j,h}\}_{j,h>0} \) defined as minimizers in \( V(\Omega, T) \) for the functionals:

\[
I_{j,h}(u) = \mathcal{H}^{n-1}(Su) + \frac{1}{h} \int_{\Omega} |u - u_j| \, d\mathcal{L}^n + \frac{1}{h} \int_{\Omega} |u - u_{j-1,h}| \, \text{Dist} (\cdot, Su_j) \, d\mathcal{L}^n,
\]

where

\[
\text{Dist} (x, A) := \begin{cases} 
\frac{1}{\text{diam } \Omega} \inf_{\xi \in A} |x - \xi| & \text{if } A \neq \emptyset; \\
1 & \text{if } A = \emptyset
\end{cases}
\]

for a subset \( A \) of \( \Omega \). More precisely, we define for any fixed positive number \( h \) the sequence of minimizers \( \{u_{j,h}\}_{j=0}^\infty \) in the following manner: Beginning with \( u_{0,h} = u_0 \), we let \( u_{1,h} \) be a minimizer of \( I_{1,h} \) in \( V(\Omega, T) \). We shall inductively determine \( u_{j,h} \), \( j = 2, 3, \ldots \), as a minimizer of \( I_{j,h} \) in \( V(\Omega, T) \), where \( u_{j-1,h} \) in the functional \( I_{j,h} \) is the minimizer given in the last step.

Our aim of this paper consists in constructing a ‘time-evolutional function’

\[
(u(t))(\cdot) = \lim_{h \to 0} u_{[t/h],h}(\cdot) \quad \text{in } \Omega \quad \text{for each } t > 0
\]

if it exists, where \([\cdot]\) is the Gaussian symbol. The main result is Theorem 3.3, which asserts that this construction is possible to be established in the sense of the convergence with respect to \( L^1(\Omega) \)-norm. The main tool to guarantee the convergence is the uniform lower density ratio bound of the discontinuity sets \( S_{u_j,h} \), which is to be proved in Lemma 2.1. The essential part of the proof is largely based on [DCL]. Upon having obtained such a uniform estimate, we shall arrive at our goal according to the method presented in the paper [ATW].

The above method for constructing a time-evolutional function was introduced by DeGiorgi in his thesis [DG]. In case \( T = \{0, 1\} \), our problem reduces to mean-curvature flow problem. Almgren, Taylor and Wang have dealt with such type of problem by making use of the notion of rectifiable current, and obtained some regularity results on time-evolutional current which corresponds to \( S_{u(t)} \) in our situation. From the result of this paper, we have not got as yet any significance which \( u(t) \) has. However, by developing our investigation for time-evolutional function it is expected to be derived explicitly a minimizing movement of such singular points as triple points (see [MS, Page 598]).
1. The construction of $h$-discrete minimizing movement

In this section, we construct a sequence of minimizers $\{u_{j,h}\}_{j=1,2,\ldots}$, and show a uniform boundedness of them.

Throughout this section we denote by $m_1$ the smallest number belonging to $T$ such that $m_1 \geq \max \{\text{sup}_\Omega u_0, \text{sup}_\Omega g\}$, and by $m_0$ the largest one belonging to $T$ such that $m_0 \leq \max \{\text{inf}_\Omega u_0, \text{inf}_\Omega g\}$. Other notations shall be as described in the introduction.

We begin with the fundamental existence theorem:

1.1. Theorem. There exists a minimizer in $V(\Omega, T)$ for the functional

$$I_1(u) = \mathcal{H}^{n-1}(S_u) + \int_\Omega |u - g|^2 d\mathcal{L}^n + \frac{1}{h} \int_\Omega |u - u_0|^2 \text{Dist}(\cdot, S_{u_0}) d\mathcal{L}^n,$$

where $h$ is a positive constant.

Proof. Let $\{u^{(k)}\}_{k=1}^\infty \subset V(\Omega, T)$ be a minimizing sequence:

$$\lim_{k \to \infty} I_1(u^{(k)}) = \inf_{V(\Omega, T)} I_1. \quad (1.1)$$

If we set $\overline{u}^{(k)} = \max \{m_0, \min \{u^{(k)}, m_1\}\}$, then $\overline{u}^{(k)} \in V(\Omega, T)$, and $I_1(\overline{u}^{(k)}) \leq I_1(u^{(k)})$ for each $k = 1, 2, \ldots$. We can thus assume without loss of generality that $u^{(k)}(x) \in T \cap [m_0, m_1]$ for $\mathcal{L}^n$-almost all $x \in \Omega$, $k = 1, 2, \ldots$. (1.2)

Then since $u^{(k)} \in SBV(\Omega, T)$ and $\nabla u^{(k)} = 0$ $\mathcal{L}^n$-almost everywhere in $\Omega$ (see Theorem A1.1 of Appendix A1 for the proof of these facts), we deduce from (1.1) and (1.2) that

$$\sup_{k=1,2,\ldots} \left\{ ||\nabla u^{(k)}||_{\infty(\Omega)} + \mathcal{H}^{n-1}(S_{u^{(k)}}) + ||u^{(k)}||_{\infty(\Omega)} \right\} < \infty. \quad (1.3)$$

Here and elsewhere, we set $||f||_{\infty(\Omega)} := \text{esssup}_\Omega |f|$ for a $\mathcal{L}^n$-measurable function defined on $\Omega$. The uniform boundedness (1.3) enables us to choose a subsequence of $\{u^{(k)}\}_{k=1}^\infty$, still denoted by the same notation, and $u \in SBV(\Omega)$ such that $\lim_{k \to \infty} u^{(k)} = u$ $\mathcal{L}^n$-almost everywhere in $\Omega$. We here in particular remark that $u \in BV(\Omega, T)$ because $u \in T$ $\mathcal{L}^n$-almost everywhere in $\Omega$. The compactness property as mentioned above is proved in [Amb1, Theorem 2.1], which also claims the lower semicontinuity

$$\mathcal{H}^{n-1}(S_u \cap \Omega) \leq \lim_{k \to \infty} \mathcal{H}^{n-1}(S_{u^{(k)}} \cap \Omega). \quad (1.4)$$

From (1.2) and the boundedness of the function $\text{Dist}(\cdot, S_{u_0})$, we are able to show that the last two terms of $I_1(u^{(k)})$ converge to those of $I_1(u)$ respectively as $k \to \infty$ with the aid of the Lebesgue convergence theorem. Hence, by combining this with (1.4) we arrive at the inequality $I_1(u) \leq \lim_{k \to \infty} I_1(u^{(k)})$, which tells us that $u$ is the desired minimizer. \qed
1.2. Proposition. Let $u$ be a minimizer of the functional $I_1$ in $V(\Omega, T)$. Then $u(x)$ belongs to $T \cap [m_0, m_1]$ for $\mathcal{L}^n$-almost all $x \in \Omega$.

Proof. We first note that $ar{u} := \max (m_0, \min (u, m_1))$ belongs to the admissible function space $V(\Omega, T)$. If $\mathcal{L}^n(\{x \in \Omega \mid u(x) > m_1 \text{ or } u(x) < m_0\}) > 0$, then we have the inequality $I_1(\bar{u}) < I_1(u)$, which contradicts to the minimality of $u$.

Let us now define $u_{j,h}$. Fix a positive number $h$. We first set $u_{0,h} := u_0$ and let $u_{1,h}$ be a minimizer of $I_1$ in $V(\Omega, T)$. Next, let $u_{2,h}$ be a minimizer of $I_2$ in the same admissible function space, where $I_2$ is the functional defined by replacing $u_0$ and $S_{u_0}$ in the functional $I_1$ with $u_{1,h}$ and $S_{u_{1,h}}$ respectively. Remark here that the existence of such a minimizer is shown in the same way as the proof of Theorem 1.1 by noticing the property asserted in Proposition 1.2. In this way, we inductively define $u_{j,h}$, $j = 3, 4, \cdots$, as a minimizer $V(\Omega, T)$ for the functional

$$I_j(u) = \mathcal{H}^{n-1}(S_u) + \int_\Omega |u - g|^2 d\mathcal{L}^n + \frac{1}{h} \int_\Omega |u - u_{j-1,h}|^2 \text{Dist}(\cdot, S_{u_{j-1,h}}) d\mathcal{L}^n.$$ 

We call such a sequence $\{u_{j,h}\}_{j=0}^\infty$ as above $h$-discrete minimizing movement.

1.3. Remark. In the same way as the proof of Proposition 1.2, we obtain $u_{j,h} \in T \cap [m_0, m_1] \mathcal{L}^n$-almost everywhere in $\Omega$ for any $j = 1, 2, \cdots$ and $h > 0$. Therefore, if we set $M = \max (|m_0|, |m_1|)$, then in particular we have $||u_{j,h}||_{\mathcal{L}^\infty(\Omega)} \leq M$ for any $j = 1, 2, \cdots$ and $h > 0$.

2. Uniform Lower density ratio bound for the free discontinuity sets

Throughout this section we fix an arbitrary positive number $h$. Let $u_j := u_{j,h}$, $j = 1, 2, \cdots$, be an $h$-discrete minimizing movement constructed as in Section 1. The main result of this section is as follows:

2.1. Lemma. Let $u_j$ be such as constructed above. Then there exist positive numbers $\varepsilon$ and $\theta$ depending only on $n$ such that

$$\mathcal{H}^{n-1}(S_{u_j} \cap \overline{B}_r(x)) \geq \varepsilon r^{n-1}$$

holds for any $j = 1, 2, \cdots$, $x \in S_{u_j} \cap \partial \Omega$ and any $r \leq R_x$, where $R_x$ is a positive constant as follows:

$$R_x := \min \left( \text{dist} (x, \partial \Omega), \frac{\varepsilon \theta}{(2M)^2 \omega_n} \left(1 + \frac{1}{h}\right)^{-1} \right).$$ (2.1)
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**Proof.** Let $B_\rho$ be an $n$-dimensional ball of radius $r$ compactly contained in $\Omega$. During this proof we fix an arbitrary positive integer $j$ and we use the notations $\Phi(u_j, B_\rho)$ and $\Psi(u_j, B_\rho)$ (see Appendix 2 for their definitions). Since from Remark 1.3 $\|u_j\|_{\infty(\Omega)} \leq M$, we get

$$\Phi(u_j, B_\rho) = \inf \left\{ \mathcal{H}^{n-1}(S_v \cap \overline{B}_\rho) \mid v \in SBV(\Omega, T) \text{ with } v = u_j \text{ in } \Omega \setminus \overline{B}_\rho \text{ and } \|v\|_{\infty(\Omega)} \leq M \right\}.$$  

(2.2)

It follows from the minimality of $u_j$ for the functional $I_j$ that

$$\mathcal{H}^{n-1}(S_{u_j} \cap \overline{B}_\rho) \leq \mathcal{H}^{n-1}(S_v \cap \overline{B}_\rho) + (2M)^2 \omega_n \left(1 + \frac{1}{h}\right) \rho^n$$

holds for any $v \in SBV(\Omega, N)$ satisfying $v = u_j$ in $\Omega \setminus \overline{B}_\rho$ and $\|v\|_{\infty(\Omega)} \leq M$. Taking infimum over all $v$ as in (2.3), we obtain from (2.2)

$$\mathcal{H}^{n-1}(S_{u_j} \cap \overline{B}_\rho) \leq \Phi(u_j, B_\rho) + (2M)^2 \omega_n \left(1 + \frac{1}{h}\right) \rho^n,$$

and hence

$$\Psi(u_j, B_\rho) \leq (2M)^2 \omega_n \left(1 + \frac{1}{h}\right) \rho^n.$$  

(2.4)

Let $\varepsilon$ and $\theta$ be positive numbers as stated in Theorem A2.1 of Appendix 2, and take a ball $B_r(x)$ with the center $x \in \Omega$ and the radius $r < R_x$, where $R_x$ is as in (2.1). Putting $B_r(x)$ instead of $B_\rho$ in (2.4), we are led to

$$\lim_{r \to 0} r^{1-n} \Psi(u_j, B_r(x)) = 0$$

(2.5)

and

$$\Psi(u_j, B_r(x)) \leq \varepsilon \theta r^{n-1} \text{ for all } r < R_x.$$  

(2.6)

We define a subset of $\Omega$ as follows:

$$\Omega_1 := \left\{ x \in \Omega \mid r^{1-n} \mathcal{H}^{n-1}(S_{u_j} \cap \overline{B}_r(x)) < \varepsilon \text{ holds for some } r < R_x \right\}.$$  

Then for any $x \in \Omega_1$, there exists a positive number $r_x < R_x$ such that

$$r_x^{1-n} \mathcal{H}^{n-1}(S_{u_j} \cap \overline{B}_{r_x}(x)) < \varepsilon.$$  

(2.7)
We deduce from (2.6) that
\[ \Psi(u_j, B_r(x)) \leq \epsilon \theta r^{n-1} \quad \text{for all } r < r_x. \] (2.8)

Facts (2.5), (2.7) and (2.8) enable us to adopt Lemma A2.1 of Appendix 2, and consequently,
\[ \lim_{r \to 0} r^{1-n} \mathcal{H}^{n-1}(S_{u_j} \cap B_r(x)) = 0 \quad \text{for any } x \in \Omega_1. \]

We thus obtain the inclusion \( \Omega_1 \subset \Omega_0 \), where
\[ \Omega_0 = \left\{ x \in \Omega \mid \lim_{r \to 0} r^{1-n} \mathcal{H}^{n-1}(S_{u_j} \cap B_r(x)) = 0 \right\}. \]

Whereas, since \( \Omega_0 \subset \Omega_1 \) follows directly from their definitions, we finally achieve \( \Omega_1 = \Omega_0 \).

We can here prove that the set \( \Omega_1 \) is open. In fact, if \( x_0 \in \Omega_0 = \Omega \), then we get
\[ \lim_{r \to 0} r^{1-n} \mathcal{H}^{n-1}(S_{u_j} \cap B_r(x_0)) = 0. \] (2.9)

Since the value \( \inf_{\xi \in B_\delta(x_0)} R_\xi \) does not decrease as \( \delta \) converges to zero, there exists a positive constant \( \delta_{x_0} \) such that the inequality \( \delta_{x_0} < R_\xi \) holds for any \( \xi \in B_{\delta_{x_0}}(x_0) \). From (2.9) there exists a positive constant \( r_0 < \delta_{x_0} \) such that
\[ r_0^{1-n} \mathcal{H}^{n-1}(S_{u_j} \cap B_{r_0}(x_0)) < 2^{1-n} \epsilon. \]

Then, for each \( \xi \in B_{r_0/2}(x_0) \), we have \( r_0/2 < R_\xi \) and
\[ \frac{\mathcal{H}^{n-1}(S_{u_j} \cap \overline{B}_{r_0/2}(\xi))}{(r_0/2)^{n-1}} \leq 2^{n-1} \frac{\mathcal{H}^{n-1}(S_{u_j} \cap \overline{B}_{r_0}(x_0))}{r_0^{n-1}} < \epsilon, \]
so that \( \xi \in \Omega_1 \). We thus have \( B_{r_0/2}(x_0) \subset \Omega_1 \), from which we deduce that \( \Omega_1 \) is open.

Noticing that \( \nabla u_j = 0 \) \( \mathcal{L}^n \)-almost everywhere in \( \Omega \) (see Lemma A1.1 of Appendix 1), we infer from [DCL, Theorem 3.6] that \( S_{u_j} \cap \Omega \subset \Omega \setminus \Omega_0 \), so that \( \overline{S}_{u_j} \cap \Omega \subset \Omega \setminus \Omega_0 \), because \( \Omega \setminus \Omega_0 \) is closed in \( \Omega \). On the other hand, \( \Omega \setminus \overline{S}_{u_j} \subset \Omega_0 \) is true, because if \( x_0 \in \Omega \setminus \overline{S}_{u_j} \), then \( \mathcal{H}^{n-1}(S_{u_j} \cap \overline{B}_r(x_0)) = 0 \) holds for sufficiently small \( r \). Thus, we obtain \( \overline{S}_{u_j} \cap \Omega = \Omega \setminus \Omega_1 \), which yields the conclusion with the condition \( r < R_\xi \). Full result follows from the approximate argument. \( \square \)
3. The construction of a $L^1$-continuous minimizing movement

Let $h > 0$ and $\{u_{j,h}\}_{j=1}^{\infty}$ be an $h$-discrete minimizing movement defined as in Section 1. Our aim of this section is to construct a $L^1$-continuous minimizing movement through such $\{u_{j,h}\}_{j=1,2,\ldots}$. For this purpose, we define some values: $\overline{h} = (1+1/h)^{-1}$ and $\delta = \varepsilon \theta / ((2M)^2 \omega_n)$, where $\varepsilon$ and $\theta$ are as in Theorem A1.1 of Appendix 1, while $M$ is the positive constant as in Remark 1.3. Moreover we set

$$
\sigma = \min_{s,t \in \mathcal{T} \cap [-M,M]} |s-t|.
$$

We remark that $\sigma$ is positive because of the property of $\mathcal{T}$. We put

$$
F(u_{j,h}) \equiv \mathcal{H}^{n-1}(S_{u_{j,h}}) + \int_{\Omega} |u_{j,h} - g|^2 d\mathcal{L}^n. \text{ for } j = 1,2,\ldots.
$$

In what follows, $C = C(*)$ denotes a positive constant depending only on the quantities $*$ appearing in parentheses, particularly, not on $h$. The following lemma will play an important role in order to reach our goal:

3.1. Lemma. Let $h > 0$ and let $u_{j,h}, j = 1,2,\ldots$, be an $h$-discrete minimizing movement. Set $\Omega_h := \{x \in \Omega \mid \text{dist}(x, \partial \Omega) > \delta \overline{h}\}$, where $\delta$ and $\overline{h}$ are as indicated above. Then, for positive integers $k, l$ satisfying $k \geq l + 1 \geq 2$ and $h(k-l) < \delta^{-n-1},$

$$
\int_{\Omega_h} |u_{k,h} - u_{l,h}| d\mathcal{L}^n \leq C(1 + h)^{n+1} \{h(k-l)\}^{\frac{1}{n+1}},
$$

where $C = C(n, M, F(u_0), \text{diam } \Omega, \sigma)$.

**Proof.** For simplicity, we denote $u_j = u_{j,h}$ and $\tau_{j-1}(\cdot) = \text{Dist}(\cdot, S_{u_j})$, $j = 1,2,\ldots$. For $j = 1,2,\ldots$ and for a positive number $R$, we have

$$
\int_{\Omega_h} |u_j - u_{j-1}|^2 d\mathcal{L}^n \leq \frac{1}{R} \int_{\Omega} |u_j - u_{j-1}|^2 \tau_{j-1} d\mathcal{L}^n + \int_{\Omega_h(\tau_{j-1} < R)} |u_j - u_{j-1}|^2 d\mathcal{L}^n
$$

$$
\equiv J_{j,1} + J_{j,2},
$$

where $\Omega_h(\tau_{j-1} < R) := \{x \in \Omega_h \mid \tau_{j-1}(x) < R\}$. Let us estimate the term $J_{j,1}$: Since $u_j$ is a minimizer of the functional $I_j$, we have $I_j(u_j) \leq I_j(u_{j-1})$, which implies

$$
\frac{1}{h} \int_{\Omega} |u_j - u_{j-1}|^2 \tau_{j-1} d\mathcal{L}^n \leq F(u_{j-1}) - F(u_j).
$$

(3.2)
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This brings us to the estimation

\[ J_{j,1} \leq \left( \frac{h}{R} \right) [F(u_{j-1}) - F(u_j)]. \] (3.3)

We now turn to the term \( J_{j,2} \). Put

\[ f_j(r) := \int_{\Omega_h \cap \tau_{j-1}^{-1}(r)} |u_j - u_{j-1}|^2 dH^{n-1} \quad \text{for } r > 0. \]

Since the function \( \tau_{j-1} : \mathbb{R}^n \to \mathbb{R} \) is Lipschitzian, we can apply [Fe, Theorem 3.2.12] to obtain

\[ \frac{1}{\text{diam } \Omega} \int_{\Omega_h(\tau_{j-1} < R)} |u_j - u_{j-1}|^2 d\mathcal{L}^n = \int_0^R f_j dr \] (3.4)

and

\[ \frac{1}{\text{diam } \Omega} \int_{\Omega_h(\tau_{j-1} < R)} |u_j - u_{j-1}|^2 \tau_{j-1} d\mathcal{L}^n = \int_0^R r f_j dr, \] (3.5)

where we use the fact that \( |\text{grad } \tau_{j-1}| = 1/\text{diam } \Omega \) \( \mathcal{L}^n \)-almost everywhere in \( \Omega \) (refer to the proof of [Fe, Lemma 3.2.34]). The following formula is shown in [ATW, Step 1 of the proof of Proposition 4.1]: For each function \( \zeta : [0, R] \to [0, S] \)

\[ \int_0^R \zeta dr \leq 2^\frac{1}{2} S^\frac{1}{2} \left( \int_0^R r \zeta(r) dr \right)^\frac{1}{2}. \]

Adopting this formula as \( \zeta = f_j \) and \( S = \sup_{[0, R]} f_j \), we obtain from (3.4) and (3.5)

\[ J_{j,2} \leq 2^\frac{1}{2} (\text{diam } \Omega)^\frac{1}{2} \left( \sup_{[0, R]} f_j \right)^\frac{1}{2} \left( \int_{\Omega_h} |u_j - u_{j-1}| ^2 \tau_{j-1} d\mathcal{L}^n \right)^\frac{1}{2}. \] (3.6)

By taking advantage of (3.2) and \( ||u_j||_{\infty(\Omega)} \leq M, j = 1, 2, \ldots \), we infer from (3.6)

\[ J_{j,2} \leq 2^\frac{1}{2} M^2 (\text{diam } \Omega)^\frac{1}{2} \left( \sup_{\rho \in [0, R]} \mathcal{H}^{n-1}(\Omega_h \cap \tau_{j-1}^{-1}(\rho)) \right)^\frac{1}{2} \left[ h(F(u_{j-1}) - F(u_j)) \right]^\frac{1}{2}. \] (3.7)

Let us now estimate the value of supremum in (3.7). We now fix a positive number \( \rho > 0 \), and let \( B = \{ B_\rho(x) \mid x \in \overline{S}_{u_{j-1}} \cap \Omega_h \} \). Then, with the help of the Besicovitch covering lemma (see for instance [Zi, Theorem 1.3.5]) there exist disjointed subcollections \( B_1, \ldots, B_N \) such that

\[ \overline{S}_{u_{j-1}} \cap \Omega_h \subset \bigcup_{l=1}^N \bigcup_{l \in \mathcal{B}_l} B_l, \] (3.8)
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where $N$ is a positive integer depending only on $n$. Since $\bar{S}_{u_{j-1}} \cap \Omega_h$ is bounded and each $B_j$ is disjointed, the cardinality of each $B_j$ is finite. Thus, the subcollection $\tilde{B} = \bigcup_{i=1}^{N} B_i$ satisfies the following: $\tilde{B}$ is a finite set and no points in $\mathbb{R}^n$ is contained in more than $N$ elements of $\tilde{B}$. We are thus capable of representing as $\tilde{B} = \{\overline{B}_\rho(x_k) \mid x_k \in \bar{S}_{u_{j-1}} \cap \Omega_h, k = 1, 2, \ldots, K\}$. Then since in particular each $x_k$ belongs to $\Omega_h$, the inequality $\text{dist}(x_k, \partial \Omega) > \delta h$ must hold. Hence, from the definition of $R_x$ (see (2.1)) we have

$$R_{x_k} = \delta h \quad \text{for any } k = 1, 2, \ldots, K. \quad (3.9)$$

If we set $d = \text{diam} \Omega$, then $\tau_{j-1}(\cdot) = \frac{1}{d} \text{dist}(\cdot, S_{u_{j-1}})$. Hence, from (3.8)

$$\tau_{j-1}^{-1}(\rho) \cap \Omega_h \subset \bigcup_{k=1}^{K} \left\{ \overline{B}_{2d \rho}(x_k) \cap \tau_{j-1}^{-1}(\rho) \right\}, \quad (3.10)$$

where $\tilde{d} = \max(1, d)$. We have for each $k = 1, 2, \ldots, K$

$$\mathcal{H}^{n-1}(\overline{B}_{2d \rho}(x_k) \cap \tau_{j-1}^{-1}(\rho)) = \mathcal{H}^{n-1}(\overline{B}_{(2\tilde{d} / d) \rho d}(x_k) \cap \sigma_{j-1}^{-1}(d \rho)) \quad (\sigma_{j-1}(\cdot) \equiv \text{dist}(\cdot, S_{u_{j-1}})) \quad (3.11)$$

where the last inequality is demonstrated by a slight variation of the proof of [ATW, Proposition 4.2, Step2]. To proceed our argument we consider the following two cases: (i) $\rho \leq \delta h$ and (ii) $\rho > \delta h$.

(i) $\rho \leq \delta h$. In this case the inequality $\rho \leq R_{x_k}$ is satisfied for each $k = 1, 2, \ldots, K$, because of (3.9). So we are able to apply Lemma 2.1 to obtain

$$\mathcal{H}^{n-1}(S_{u_j} \cap \overline{B}_\rho(x_k)) \geq \varepsilon \rho^{n-1}. \quad (3.12)$$

Combining (3.11) with (3.12) and taking (3.10) into account, we deduce

$$\mathcal{H}^{n-1}(\tau_{j-1}^{-1}(\rho) \cap \Omega_h) \leq \frac{C_1(n, d)}{\varepsilon} \sum_{k=1}^{K} \mathcal{H}^{n-1}(S_{u_j} \cap \overline{B}_\rho(x_k)). \quad (3.13)$$

We here recall that $\{B_\rho(x_k)\}_{k=1}^{K}$ has at most $N = N(n)$ overlaps. Thus (3.13) implies

$$\mathcal{H}^{n-1}(\tau_{j-1}^{-1}(\rho) \cap \Omega_h) \leq \frac{C_2(n, d)}{\varepsilon} \mathcal{H}^{n-1}(S_{u_j} \cap \Omega). \quad (3.14)$$
(ii) $\rho > \delta \overline{h}$. Bearing in mind (3.9), we are able to apply Lemma 2.1 as $r = \delta \overline{h}$:

$$\mathcal{H}^{n-1}(S_{u_j} \cap \overline{B}_\rho(x_k)) \geq \mathcal{H}^{n-1}(S_{u_j} \cap \overline{B}_{\delta \overline{h}}(x_k)) \geq \{\varepsilon(\delta \overline{h}/\rho)^{n-1}\} \rho^{n-1}.$$ 

Starting from this estimate instead of (3.12), we are led to

$$\mathcal{H}^{n-1}(\tau_{j-1}^{-1}(\rho) \cap \Omega_h) \leq \frac{C_2}{\varepsilon} \left( \frac{\rho}{\delta \overline{h}} \right)^{n-1} \mathcal{H}^{n-1}(S_{u_j}).$$  \hspace{1cm} (3.15)

corresponding to (3.14).

Coupling (3.14) and (3.15), we deduce

$$\mathcal{H}^{n-1}(\tau_{j-1}^{-1}(\rho) \cap \Omega_h) \leq \frac{C_2}{\varepsilon} \max \left( 1, \left( \frac{\rho}{\delta \overline{h}} \right)^{n-1} \right) \max \left( 1, \left( \frac{\rho}{\delta \overline{h}} \right)^{n-1} \right) F(u_j)$$  \hspace{1cm} (3.16)

for any $\rho > 0$. In particular, (3.2) tells us that $F(u_j) \leq F(u_{j-1})$, $j = 1, 2, \cdots$, and hence from (3.16)

$$\mathcal{H}^{n-1}(\tau_{j-1}^{-1}(\rho) \cap \Omega_h) \leq \frac{C_2}{\varepsilon} \max \left( 1, \left( \frac{\rho}{\delta \overline{h}} \right)^{n-1} \right) F(u_0).$$

Imposed here the restriction

$$R > \delta \overline{h},$$  \hspace{1cm} (3.17)

we get

$$\max \left( 1, \left( \frac{\rho}{\delta \overline{h}} \right)^{n-1} \right) \leq \left( \frac{R}{\delta \overline{h}} \right)^{n-1} \text{ for } \rho < R.$$

Therefore

$$\sup_{[0,R]} \mathcal{H}^{n-1}(\tau_{j-1}^{-1}(\rho) \cap \Omega_h) \leq C_3 \left( \frac{R}{\overline{h}} \right)^{n-1},$$

where $C_3 = C_3(n, M, F(u_0), \text{diam} \Omega)$. We thus from (3.7)

$$J_{j,2} \leq C_4 \left( \frac{R}{\overline{h}} \right)^{\frac{n-1}{2}} \left[ h \{F(u_j) - F(u_{j-1})\} \right]^{\frac{1}{2}},$$

where $C_4 = C_4(n, M, F(u_0), \text{diam} \Omega)$. Summarizing (3.1), (3.18) and (3.19), there shall be derived the estimation

$$\int_{\Omega_h} |u_j - u_{j-1}|^2 d\mathcal{L}^n \leq$$

$$\leq \frac{h}{R} |F(u_{j-1}) - F(u_j)| + C_4 \left( \frac{R}{\overline{h}} \right)^{\frac{n-1}{2}} \left[ h \{F(u_{j-1}) - F(u_j)\} \right]^{\frac{1}{2}}.$$  \hspace{1cm} (3.20)
Noticing that the value \(|u_j - u_{j-1}|\) takes zero or positive number greater than or equal to \(\sigma\) in \(\Omega\), we have

\[
\sigma \int_{\Omega_h} |u_j - u_{j-1}| d\mathcal{L}^n \leq \int_{\Omega_h} |u_j - u_{j-1}|^2 d\mathcal{L}^n.
\]

Hence, in light of (3.20)

\[
\int_{\Omega_h} |u_j - u_{j-1}| d\mathcal{L}^n \leq \sigma \left\{ \frac{h}{R} [F(u_{j-1}) - F(u_j)] + C_4 \left( \frac{R}{h} \right)^{\frac{n-1}{2}} [h(F(u_{j-1}) - F(u_j))^{\frac{1}{2}}] \right\}.
\]

We carry out a summation of inequalities (3.21) over \(j\) from \(l + 1\) to \(k\) with \(k \geq l + 1 \geq 2\), so that

\[
\int_{\Omega_h} |u_k - u_l| d\mathcal{L}^n \leq \sigma \left\{ \left( \frac{h}{R} \right) F(u_0) + C_4 \left( \frac{R}{h} \right)^{\frac{n-1}{2}} \{h(k-l)\}^{\frac{1}{2}} F(u_0)^{\frac{1}{2}} \right\}
\]

\[
= C_5 \left\{ \left( \frac{h}{R} \right) + \left( \frac{R}{h} \right)^{\frac{n-1}{2}} \{h(k-l)\}^{\frac{1}{2}} \right\},
\]

where \(C_5 = C_5(n, M, F(u_0), \text{diam } \Omega, \sigma)\). Set here \(R = h \{h(k-l)\}^{1/n+1}\). Then owing to the assumption \(h(k-l) < \delta^{-n-1}\) we have \(\delta h < R\), which assures the restriction (3.17). Hence, we can conclude the claim as required.

We are now in a position to construct a \(L^1\)-continuous minimizing movement. Let \(h > 0\) and \(\{u_{j,h}\}_{j=0}^{\infty}\) be the \(h\)-discrete minimizing movement. We first define an approximate minimizing movement as follows:

\[
u_{h}(t) := u_{[t/h],h} \quad \text{in } \Omega \quad \text{for } t \geq 0,
\]

where \([\cdot]\) denotes the Gaussian symbol. Then the following holds:

3.2. Proposition. There exists a subsequence \(\{u_{h_k}\}_{k=1}^{\infty}\) of \(\{u_{h}\}_{h>0}\) such that the following holds: For any \(s \in Q^+\) there exists \(u(s) \in V(\Omega, T)\) such that

\[
\lim_{k \to \infty} u_{h_k}(s) = u(s) \quad \text{in } L^1(\Omega).
\]
is fulfilled for each \( s \in \mathbb{Q}^+ \).

**Proof.** By means of (3.2) we in particular have \( F(u_j) \leq F(u_{j-1}) \), \( j = 1, 2, \ldots \), and hence \( F(u_j) \leq F(u_0) \), \( j = 1, 2, \ldots \). Therefore, \( F(u_h(s)) \leq F(u_0) \) holds for each \( s \in \mathbb{Q}^+ \). Moreover by Remark 1.3 we have \( ||u_h(s)||_{\infty(\Omega)} \leq M \) for each \( s \in \mathbb{Q}^+ \). Thus, in view of the compactness result [Amb1, Theorem 2.1] and the diagonal process, we arrive at the conclusion.

The following theorem is the aim of this paper:

**3.3. Theorem.** There exists a sequence \( h_k \downarrow 0 \) as \( k \to \infty \) and \( u(t) \in V(\Omega, T) \) for each \( t > 0 \) such that

\[
\lim_{k \to \infty} u_{h_k}(t) = u(t) \quad \text{in } L^1(\Omega). \tag{3.22}
\]

The convergence of (3.22) is locally uniform in \( t \in [0, T] \) for any \( T > 0 \). Furthermore,

\[
||u(t) - u(s)||_{L^1(\Omega)} \leq C|s - t|^\frac{n+1}{n+1}
\]

is fulfilled for any \( s, t > 0 \) with \( |s - t| < \delta^{-n-1} \), where \( C = C(n, F(u_0), \text{diam } \Omega, \sigma) \).

**Proof.** Let \( \tilde{\Omega} \) be a subdomain compactly contained in \( \Omega \), and

\[
h_0 := \left\{ \max \left( \frac{\delta}{\text{dist}(\tilde{\Omega}, \Omega)} - 1, 1 \right) \right\}^{-1}.
\]

Remark that \( \tilde{\Omega} \subset \Omega_h \) for any \( h < h_0 \). By Lemma 3.1, if positive numbers \( s \) and \( t \) satisfy \( |s - t| < \delta^{-1-n}/2 \), then

\[
\int_{\tilde{\Omega}} |u_h(s) - u_h(t)|d\mathcal{L}^n = \int_{\tilde{\Omega}} |u_{[\frac{s}{h}],h} - u_{[\frac{t}{h}],h}|d\mathcal{L}^n
\]

\[
\leq C(1 + h)^{n-1} \left\{ h \left( \left\lfloor \frac{s}{h} \right\rfloor - \left\lfloor \frac{t}{h} \right\rfloor \right) \right\}^{\frac{1}{n+1}} \leq C'(|s - t| + h)^{\frac{1}{n+1}}
\]

is valid for \( h < \min(1, h_0, \delta^{-n-1}/2) \), where \( C' = C'(n, M, F(u_0), \text{diam } \Omega, \sigma) \). Applying Proposition 3.2 to the similar argument as in the proof of Ascoli-Alzerá theorem, we accomplish that there exists \( u(s) \in V(\tilde{\Omega}, T) \) for each \( s > 0 \) such that

\[
\int_{\tilde{\Omega}} |u(s) - u(t)|d\mathcal{L}^n \leq C'|s - t|^{\frac{1}{n+1}}.
\]

Finally if we remark that the constant \( C' \) is independent of the choice of \( \tilde{\Omega} \), we arrive at the desired result.

\( \square \)
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Appendix 1. Remark on $SBV(\Omega, T)$-function

Let $T$ be the subset of $\mathbb{R}^n$ as defined in the introduction. Then, since $T$ is at most countable, it is enough to treat the case where we are able to write $T = \{\alpha_j\}_{j \in \mathbb{Z}}$ such that $\alpha_j < \alpha_{j+1}$ holds for each $j \in \mathbb{Z}$. Hereafter we denote by $\mathbb{Z}$ the set of all integers.

A1.1. Theorem. Let $\Omega$ be a domain in $\mathbb{R}^n$ and set $BV(\Omega, T) := \{u \in BV(\Omega) | u(x) \in T \text{ for } \mathcal{L}^n\text{-almost all } x \in \Omega\}$. Then $BV(\Omega, T) = SBV(\Omega, T)$. Moreover, $u \in BV(\Omega, T)$ satisfies $\nabla u = 0 \mathcal{L}^n$-almost everywhere in $\Omega$, where $\nabla u$ is the approximate differential of $u$.

Proof. Since $SBV(\Omega) \subset BV(\Omega)$, in order to show the first assertion we need only to show $BV(\Omega, T) \subset SBV(\Omega, T)$. We suppose $u \in BV(\Omega, T)$. Then, noticing $u(x) \in T$ for $\mathcal{L}^n$-almost all $x \in \Omega$, we infer from co-area formula for $BV$-functions (see [Zi, Theorem 5.4.4])

$$
\int_{\Omega} |Du| = \sum_{j \in \mathbb{Z}} \int_{\Omega} |D\chi_{\{u > \alpha_j\}}|,
$$

where $\Omega(u > \alpha_j) := \{x \in \Omega | u(x) > \alpha_j\}$ and $\chi_{\{u > \alpha_j\}}$ is its characteristic function. Suppose $\partial_M \Omega(u > \alpha_j)$ to be the measure theoretic boundary of the set $\Omega(u > \alpha_j)$ (see [Zi, Definition 5.8.4]). Then, by [Zi, Theorem 5.8.1 and Lemma 5.9.5] (A1.1) leads to

$$
\int_{\Omega} |Du| = \sum_{j \in \mathbb{Z}} \mathcal{H}^{n-1}(\Omega \cap \partial_M \Omega(u > \alpha_j)).
$$

Let us now prove

$$
S_u = \bigcup_{j \in \mathbb{Z}} (\Omega \cap \partial_M \Omega(u > \alpha_j)).
$$

If $x_0 \in \bigcap_{j \in \mathbb{Z}} (\Omega \cap \partial_M \Omega(u > \alpha_j))^c$, then we get $\lim_{\rho \to 0} \rho^{-n} \mathcal{L}^n(B_\rho(x_0)(u \neq \alpha_{k_0})) = 0$ for some $k_0 \in \mathbb{Z}$ or $\lim_{\rho \to 0} \rho^{-n} \mathcal{L}^n(B_\rho(x_0)(u \leq \alpha_k)) = 0$ for any $k \in \mathbb{Z}$. This means that the approximate limit of $u$ at $x_0$ exists and is equal to $\alpha_{k_0}$ or $+\infty$ respectively. In any case we have $x_0 \notin S_u$. Moreover, if $x_0 \in \bigcup_{j \in \mathbb{Z}} (\Omega \cap \partial_M \Omega(u > \alpha_j))$, then there exists $j_0 \in \mathbb{Z}$ such that $x_0 \in \Omega \cap \partial_M \Omega(u > \alpha_{j_0})$. By definition

$$
0 < \lim_{\rho \to 0} \frac{\mathcal{L}^n(B_\rho(x_0)(u > \alpha_{j_0}))}{\rho^n} \quad \text{and} \quad 0 < \lim_{\rho \to 0} \frac{\mathcal{L}^n(B_\rho(x_0)(u \leq \alpha_{j_0}))}{\rho^n}.
$$

Recalling that $u(x) \in T$ for $\mathcal{L}^n$-almost all $x \in \Omega$, we deduce from (A1.4) that

$$
0 < \lim_{\rho \to 0} \frac{\mathcal{L}^n(B_\rho(x_0)(u > \alpha_{j_0} + \delta_+))}{\rho^n} \quad \text{and} \quad 0 < \lim_{\rho \to 0} \frac{\mathcal{L}^n(B_\rho(x_0)(u < \alpha_{j_0} + \delta_-))}{\rho^n}.
$$
for any $\delta_+ \in (0, \alpha_{j_0+1} - \alpha_{j_0})$ and $\delta_- \in (0, \alpha_{j_0} - \alpha_{j_0-1})$. Denote the approximate upper and lower limit of $u$ at $x_0$ by $u^+(x_0)$ and $u^-(x_0)$ respectively. Then from the definition $u^+(x_0) \geq \alpha_{j_0} + \delta_+$ and $u^-(x_0) \leq \alpha_{j_0} + \delta_-$ holds for any $\delta_+ \in (0, \alpha_{j_0+1} - \alpha_{j_0})$ and $\delta_- \in (0, \alpha_{j_0} - \alpha_{j_0-1})$ by means of (A1.5) (see [Amb2]). This yields $u^+(x_0) \geq \alpha_{j_0} + |\alpha_{j_0+1} - \alpha_{j_0}|$ and $u^-(x_0) \leq \alpha_{j_0}$, and therefore we have $x_0 \in S_u$. Thus (A1.3) follows.

From (A1.2) and (A1.3) we obtain $|Du|(\Omega) = |Du|(S_u)$. We thus have $|Cu|(\Omega) = 0$ and $\int_{\Omega} |\nabla u| d\mathcal{L}^n = 0$, where $|Cu|$ is the total variation of the Cantor part of the measure $Du$ (see (3.1) and (3.2) of [Amb1]). In particular, $|Cu|(\Omega) = 0$ implies that $u \in SBV(\Omega, T)$.

Thanks to the above argument, $\int_{\Omega} |\nabla u| d\mathcal{L}^n = 0$ holds for any $u \in BV(\Omega, T)$, which achieves the second half of Theorem.

**Appendix 2. Density estimate for quasi-minima in $SBV(\Omega, T)$**

The results of this appendix has already included in [DCL]. However, for reader's convenience, we shall describe the outline of the proofs restricted to our case.

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, and let $K$ be a closed subset of $\Omega$. For $u \in SBV(\Omega, T)$, we set

$$
\Phi(u, K) = \inf \{\mathcal{H}^{n-1}(S_v \cap K) \mid v \in SBV(\Omega, T) \text{ with } v = u \text{ in } \Omega \setminus K\};
$$

$$
\Psi(u, K) = \mathcal{H}^{n-1}(S_u \cap K) - \Phi(u, K).
$$

Our goal of this appendix is to prove the following:

**A2.1. Theorem.** There exist positive numbers $\varepsilon$ and $\theta$, depending only on $n$, such that

$$
\lim_{\rho \to 0} \rho^{1-n} \mathcal{H}^{n-1}(S_u \cap \overline{B}_\rho(x_0)) = 0,
$$

provided

(a) $\mathcal{H}^{n-1}(S_u \cap \overline{B}_R(x_0)) < \varepsilon R^{n-1}$;

(b) $\Psi(u, \overline{B}_\rho(x_0)) \leq \varepsilon \rho^{n-1}$ for $\rho < R$;

(c) $\lim_{\rho \to 0} \rho^{1-n} \Psi(u, \overline{B}_\rho(x_0)) = 0$

for some $x_0 \in \Omega$, $R < \text{dist}(x_0, \partial \Omega)$ and $u \in SBV(\Omega, T)$.

Theorem A2.1 follows from the following lemma through a certain iteration argument based on the following lemma (refer to the proofs of [DCL, Lemma 4.9 and 4.10]):

**A2.2. Lemma.** For any $\alpha \in (0, 1)$ and $\delta > 0$, there exist $\bar{\varepsilon} = \bar{\varepsilon}(n, \alpha, \delta) > 0$ and $\bar{\theta} = \bar{\theta}(n, \alpha, \delta) > 0$ such that

$$
\mathcal{H}^{n-1}(S_u \cap \overline{B}_{\alpha \rho}(x_0)) \leq \delta \mathcal{H}^{n-1}(S_u \cap \overline{B}_\rho(x_0)),
$$

for any $\rho \in (0, R)$ and $x_0 \in \Omega$.
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provided

(a) \( \mathcal{H}^{n-1}(S_u \cap \overline{B}_\rho(x_0)) < \bar{\rho}^{n-1} \);

(b) \( \Psi(u, \overline{B}_\rho(x_0)) \leq \bar{\theta} \mathcal{H}^{n-1}(S_u \cap \overline{B}_\rho(x_0)) \)

for some \( x_0 \in \Omega, \rho < \text{dist} (x_0, \partial\Omega) \) and \( u \in SBV(\Omega, T) \).

Before showing Lemma A2.2 we make some preparations. Let \( B \) be an \( n \)-dimensional ball in \( \mathbb{R}^n \). For \( u \in SBV(B, \mathbb{R}) \) satisfying \( 0 < \mathcal{H}^{n-1}(S_u \cap B) < (2\gamma_n)^{-n/n-1} \), where \( \gamma_n \) is the isoperimetric constant, we set

\[
\tau'' = \tau''(u, B) = \inf \{ t \in \mathbb{R} : \mathcal{L}^n(B(u < t)) \geq \mathcal{L}^n(B) - \frac{2\gamma_n \mathcal{H}^{n-1}(S_u \cap B)}{\mathcal{L}^n(B)/2} \}
\]

\[
m = m(u, B) = \inf \{ t \in \mathbb{R} : \mathcal{L}^n(B(u < t)) \geq \frac{\mathcal{L}^n(B)}{2} \}
\]

\[
\tau' = \tau'(u, B) = \inf \{ t \in \mathbb{R} : \mathcal{L}^n(B(u < t)) \geq \frac{2\gamma_n \mathcal{H}^{n-1}(S_u \cap B)}{\mathcal{L}^n(B)/2} \}.
\]

We here remark that \( \tau' \leq m \leq \tau'' \) always holds. We now assert the special case of [DCL, Theorem 3.1] as follows:

A2.3. Proposition. Let \( u \in SBV(\Omega, T) \) satisfying \( 0 < \mathcal{H}^{n-1}(S_u \cap B) < (2\gamma_n)^{-n/n-1} \) \( \{\mathcal{L}^n(B)/2\}^{n/n-1} \). Then, \( \tau' = m = \tau'' \) holds, and the value belongs to \( T \).

Proof. Let us prove the equalities \( \tau' = m = \tau'' \). Suppose on the contrary, for example, that \( \tau' < m = \tau'' \). Set \( \bar{u} = \min (\max (u, \tau'), \tau'') \). Since \( \bar{u} \in SBV(B, T) \), \( \nabla \bar{u} = 0 \) \( \mathcal{L}^n \)-almost everywhere in \( B \) (see Theorem A1.1 of Appendix1). Hence, by (3.3) in the proof of [DCL, Theorem 3.1] \( \int_B |D\bar{u}| = 0 \). This yields that \( \bar{u} = C \) holds for some constant \( C \in [\tau', \tau''] \). If \( C \in (\tau', \tau'') \), then \( u \equiv C \) in \( B \) and so \( S_u = \phi \), which contradicts to the assumption \( \mathcal{H}^{n-1}(S_u \cap B) > 0 \). We next suppose that \( C = \tau' \). Then \( u \leq \tau' < \tau'' \) in \( B \), which contradicts to the definition of \( \tau'' \). A similar contradiction is reached if we suppose \( C = \tau'' \). Thus the first assertion of Proposition is accomplished. The second one is concluded in view of the fact that \( u \in T \) \( \mathcal{L}^n \)-almost everywhere in \( \Omega \).

The assertion of the following lemma has been used in the proof of [DCL, Theorem 4.8]. However, it is worth while restating in the form applicable to only \( SBV(\Omega, T) \)-functions.

A2.4. Lemma. Let \( B_R \subset \mathbb{R}^n \) be a ball of radius \( R > 0 \), and let \( B_\rho \), \( 0 < \rho < R \), be the ball concentric with \( B_R \) of radius \( \rho \). Suppose \( \{\varepsilon_j\}_{j=1}^\infty \) to be a sequence of positive numbers such that \( \lim_{j \to \infty} \varepsilon_j = 0 \). Then, if \( u_j \in SBV(\Omega, T) \) satisfies \( \mathcal{H}^{n-1}(S_{u_j} \cap B_R) > 0 \) and

\[
\lim_{j \to \infty} \frac{1}{\varepsilon_j} \mathcal{H}^{n-1}(S_{u_j} \cap B_R) < \infty,
\]

(A2.1)
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then

\[ \lim_{j \to \infty} \frac{1}{\varepsilon_j} \mathcal{H}^{n-1}(\partial B_\rho \cap \{u_j \neq m(u_j, B_R)\}) = 0 \]

is fulfilled for almost all \( \rho \in (0, R) \).

**Proof.** We first remark that

\[ \int_0^R \frac{1}{\varepsilon_j} \mathcal{H}^{n-1}(\partial B_\rho \cap \{u_j \neq m(u_j, B_R)\})d\rho = \frac{1}{\varepsilon_j} \mathcal{L}^n(B_R \cap \{u_j \neq m(u_j, B_R)\}). \quad (A2.2) \]

On account of the assumption (A2.1), for sufficient large \( j \), \( u_j \) satisfies the hypothesis of Proposition A2.3. Thereby \( \tau'_j = \tau''_j = m(u_j, B_R) \) turns out to be valid, where \( \tau'_j = \tau'_j(u_j, B_R) \) and \( \tau''_j = \tau''_j(u_j, B_R) \). Hence, for such \( j \), \( B_R(u_j \neq m(u_j, B_R)) = B_R(u_j > \tau''_j) \cup B(u_j < \tau'_j) \), and so we infer from definitions of \( \tau'_j \) and \( \tau''_j \) that

\[ \mathcal{L}^n(B_R \cap \{u_j \neq m(u_j, B_R)\}) \leq 2(2\gamma_n \mathcal{H}^{n-1}(S_{u_j} \cap B_R))^{\frac{n}{n-1}}. \quad (A2.3) \]

Consequently, (A2.2) and (A2.3) yield

\[ \int_0^R \frac{1}{\varepsilon_j} \mathcal{H}^{n-1}(\partial B_\rho \cap \{u_j \neq m(u_j, B_R)\})d\rho \leq \]

\[ \leq 2(2\gamma_n)^{\frac{n}{n-1}} \left( \frac{1}{\varepsilon_j} \right)^{\frac{n}{n-1}} \left\{ \frac{1}{\varepsilon_j} \mathcal{H}^{n-1}(S_{u_j} \cap B_R) \right\}^{\frac{n}{n-1}} \quad (A2.4) \]

By recalling (A2.1) and the assumptions that \( \lim_{j \to \infty} \varepsilon_j = 0 \), the left side of (A2.4) vanishes as \( j \to \infty \), from which the desired result follows. \( \square \)

**Proof of Lemma A2.2.** If \( \mathcal{H}^{n-1}(S_u \cap B_\rho(x_0)) = 0 \), then the conclusion directly follows. Therefore, we assume \( \mathcal{H}^{n-1}(S_u \cap B_\rho(x_0)) > 0 \). By the change of variables \( \xi = \rho(x - x_0) \) and a contradiction argument, the fact we have to show is described as follows: We suppose \( \{v_j\}_{j=1}^\infty \subset SBV(B_1(0), \mathcal{T}) \) and sequences of positive numbers \( \{\theta_j\}_{j=1}^\infty, \{\varepsilon_j\}_{j=1}^\infty \) satisfy \( \lim_{j \to \infty} \varepsilon_j = \lim_{j \to \infty} \theta_j = 0 \) and

(a') \( \mathcal{H}^{n-1}(S_{v_j} \cap B_1(0)) > 0 \) and \( \frac{1}{\varepsilon_j} \mathcal{H}^{n-1}(S_{v_j} \cap B_1(0)) = 1 \);

(b') \( \frac{1}{\varepsilon_j} \Psi(v_j, \overline{B}_1(0)) \leq \theta_j \).

Then for any \( \alpha < 1 \), there exists a positive integer \( j_0 \) such that

\[ \frac{1}{\varepsilon_j} \mathcal{H}^{n-1}(S_{v_j} \cap \overline{B}_\alpha(0)) \leq \delta \quad (A2.5) \]

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holds for any \( j \geq j_0 \).

From now on, we shall prove (A2.5). Let \( \rho \in (0,1) \). We set:

\[
\tilde{v}_j := \begin{cases} 
0 & \text{in } B_r(0); \\
v_j - m(v_j, B_1(0)) & \text{in } B_r(0)^c.
\end{cases}
\]

As stated in the proof of Proposition A2.3 we have \( m(v_j, B_1(0)) \in T \) in \( \Omega \). Hence, \( \tilde{v}_j \in SBV(B_r(0), T) \). Thus, by the definition of \( \Phi \) we have

\[
\frac{1}{\varepsilon_j} \Phi(v_j, B_r(0)) = \frac{1}{\varepsilon_j} \Phi(v_j - m(v_j, B_1), B_r(0)) \leq \frac{1}{\varepsilon_j} \mathcal{H}^{n-1}(S_{\tilde{v}_j} \cap B_r(0)). \tag{A2.6}
\]

By noticing that \( S_{\tilde{v}_j} \cap B_r(0) = \partial B_r(0) \cap \{ v_j \neq m(v_j, B_1(0)) \} \) and the assumption \( (b') \), we infer from (A2.6) that

\[
\lim_{j \to \infty} \frac{1}{\varepsilon_j} \mathcal{H}^{n-1}(S_{\tilde{v}_j} \cap B_r(0)) \leq \lim_{j \to \infty} \frac{1}{\varepsilon_j} \mathcal{H}^{n-1}(\partial B_r(0) \cap \{ v_j \neq m(v_j, B_1(0)) \}). \tag{A2.7}
\]

From the assumption \( (a') \), the hypothesis of Lemma A2.4 with \( u_j \) replaced by \( v_j \) satisfies. Hence, the right side of (A2.7) is equal to zero for almost all \( r \in (0,1) \). Thus, the conclusion (A2.5) follows.

\[\square\]