On the Turing Machine with Linear-Time Transition Function

Masamichi Wate
(Received October 31, 1990)

We consider a Turing machine with countably infinite working tapes. Each tape has a head, but the heads of only finite tapes can access simultaneously. The black box is not a finite control, and the amount of informations of the transition function increases within a linear function as a function of the predecessor. Then the languages accepted within polynomial time by new type machines coincide with them accepted within exponential time by usual type machines.

§ 0 Introduction

We consider a Turing machine with a one-way infinitely long input tape and countably many one-way infinitely long working tapes. Each tape of this machine has a head, but only finite heads of these can scan simultaneously at each step. The number of scanning heads starts from one, namely only the head of the input tape, and increases step by step according to the transition function of this machine. The transition function is a function to determine the next state, the rewritten symbols and the directions of the moved heads from a current state and scanned symbols at each time. We assume that this function is linear-time computable, i.e., this is computable within linear-time of the size of arguments by some Turing machine. We say such a machine deterministic or non-deterministic linear-time Turing machine, according as its transition function is one-valued or many-valued.

Formally, this is

\[ M = <Q, \Sigma, \Gamma, B, \delta, q_0, q_A, q_B>, \]

where \( Q \) is a finite set of states, \( \Sigma \) is an input alphabet, \( \Gamma (\geq \Sigma) \) is a working alphabet, \( B (\in \Gamma - \Sigma) \) is a blank symbol,

\[ \delta : (Q - \{q_A, q_R\}) \times \Sigma \times \Gamma^* \longrightarrow Q \times \{C, R\} \times (\Gamma \times \{L, C, R\})^*. \]

and \( q_0, q_A, q_R \) are initial, accepting, rejecting states, respectively, and moreover \( L, C, R, \) * in \( \delta \) mean left, center, right, Kleene star, respectively.

The purpose of this paper is to prove the next theorem:

**Theorem** Let \( f \) be a function such that \( f(x) \geq O(x) \). Then the class of the languages accepted by deterministic (or non-deterministic) linear-time Turing machine within time \( O(f(|w|)) \) coincides with the class of the languages accepted by usual one within time \( O(2^{cf(|w|)}) \), where \( c \) is a constant and \( w \) is an input string.

The proof of this theorem follows immediately from Lemmas in the next two sections.
The first Lemma was talked in “International Symposium on Mathematical Logic and its Applications” at November 1988, and the second Lemma was talked in “The Fifth Meeting of Symbolic Logic and Information Science” which was partially supposed be Grant-in-Aid for Co-operative Research (No. 61302010), the Ministry of Education, Science and Culture at January 1989. Since the condition for a transition function was too strong in the former papers [1] and [2], it is weakened in this paper. But the technique of the proof depends on them partially. We thank for good advices by participants of two symposia.

§ 1 The simulation of a linear-time Turing machine by usual one

Lemma Suppose that a deterministic (or non-deterministic) linear-time Turing machine $M$ accepts or rejects a string $w$ within time $f(|w|)$, where $f$ is a function described in Theorem. Then there is a usual one $M'$ and a constant $c$ which accepts or rejects the same string $w$ within time $O(2cf(|w|))$.

Proof Since $\delta$ is linear-time computable, $|\delta(q, a_0, a_1, \ldots, a_n)| \leq a(n+2) + b = an + (2a + b)$, where $a$ and $b$ are constants. If we rewrite $2a + b$ as $b$, we may put $|\delta(q, a_0, a_1, \ldots, a_n)| \leq an + b$, where $a > 1$ and $b \geq 2$. Arguments $a_0, a_1, \ldots, a_n$ of $\delta$ are meant the scanned symbols of the input tape, the 1st, \ldots, the $n$-th working tapes, respectively.

At the 0-th step, $n=0$. So, the number of the scanning heads is at most $b$, after one step. So, at the 1st step, $n=b$. Similarly, at the 2nd step, $n \leq ab + b$. At the 3rd step, $n \leq a(ab + b) + b = a^2b + ab + b$. \ldots At the $t$-th step, $n \leq a^{t-1}b + a^{t-2}b + \ldots + b = a^t - 1 - 1 \leq Aa^t$, where $A = \frac{b}{a-1}$. (This is reasonable, since $a > 1$.) We simulate this machine by a usual Turing machine $M'=<Q', \Sigma, \Gamma', B, \delta', q_0, q_A, q_B>$. This machine has a one-way infinitely long input tape and three one-way infinitely long working tapes. Therefore $\delta'$ is $\delta': (Q' \times \{q_A, q_B\}) \times \Sigma \times \Gamma'^3 \longrightarrow Q' \times \{(C, R) \times (\Gamma' \times (L, C, R))^3\}$, where $Q'$ and $\Gamma'$ are chosen appropriately. The first working tape of $M'$ is provided for simulating all the working tapes of $M$ by itself alone, the second of it is done for computing the value of $\delta$, and the third of it is done for counting the number of steps $t$, in the other word, for playing a role of a clock. The $j$-th square of the $i$-th working tape of $M$ corresponds to the $\frac{1}{2}(i+j-2)(i+i-1)+j$-th square of the first working tape of $M'$. The already referred squares of this tape are divided moreover into two tracks. The upper track is provided for printing the same symbols as them on the corresponding squares of $M$, and the lower track is done for knowing the squares corresponding to the head positions of $M$. In Figure 2, each $\$ represents the leftmost square of its tape, each $\uparrow$ does the square of a head position of $M$, and each number $i$ under the first working tape does a square of the $i$-th working tape of $M$.

Now, after $t$ steps in $M$, the number of the scanned working tapes is at most $Aa^t$, and the number of the scanned squares in each working tape is at most $t$. So, for example, if the situation of $M$ is as in Figure 1 after $t$ steps, then the corresponding one of $M'$ is as in Figure 2.
On the Turing Machine with Linear-Time Transition Function

Then, we consider the move of $M$ from the $t$-th step to the $t+1$-st step. $M'$ spends $2(t+1)$ steps to count $t$ of the third working tape and to return back to the leftmost square. Next, $M'$ spends at most $2Aa^{t+1}$ steps to compute the value of $\delta$ on the second working tape and to return back to the leftmost square. Last, $M'$ spends at most $(Aa^{t+1} + t)(Aa^{t+1} + t + 1) + 2Aa^{t+1}$ steps to simulate one working tape of $M$ on the first working tape of $M'$, and the number of tapes to simulate is at most $Aa^{t+1}$, and therefore $M'$ spends at most $Aa^{t+1}[ (Aa^{t+1} + t)(Aa^{t+1} + t + 1) + 2Aa^{t+1} ]$ steps as a whole. Therefore, $M'$ only needs at most $O(a^{3t})$ steps to simulate one step of $M$ to the $t+1$-st step from the $t$-th step. Therefore, when $M$ accepts or rejects some string $w$ within $t$ step, $M'$ does the same string $w$ within $O(\sum_{i=0}^{t} a^{3i})$ steps. But $\sum_{i=0}^{t} a^{3i} = \frac{a^{3(t+1)} - 1}{a^3 - 1}$, so $O(\sum_{i=0}^{t} a^{3i}) = O(a^{3t})$. Hence, if $t = f(|w|)$, $M'$ spends $O(a^{3f(|w|)})$ steps to simulate $M$. So, if we put $c = 3\log_2 a$, then $M'$ spends $O(2e^{f(|w|)})$ steps to do it.
§ 2 The simulation of a usual Turing machine by a linear-time one

Lemma Suppose that a usual deterministic (or non-deterministic) Turing machine $M$ accepts or rejects some string $w$ within time $f(|w|)$, where $f(x) \geq O(2^x)$ for some constant $c$. Then there is a linear-time one $M'$ which accepts or rejects the same string $w$ within time $O(\log f(|w|))$.

Proof Let $M$ have $m$ working tapes. Let $M = \langle Q, \Sigma, \Gamma, B, \delta, q_0, q_A, q_R \rangle$ accept or reject a string $w$ within $f(|w|)$ steps. We may put $g(x) = \log_2 f(x)$, since $f(x) \neq 0$ except finite arguments. Then $f(|w|) = 2^{g(|w|)}$. Then, the number of scanned squares is at most $2^{g(|w|)}$ for each working tape. Let us construct a linear-time Turing machine $M' = \langle Q', \Sigma, \Gamma', B, \delta', q_0, q_A, q_R \rangle$ simulating $M$, where $\Gamma' = \Gamma \cup \{a_h | a \in \Gamma \}$ and $Q'$ is chosen appropriately.

In each working tape of $M'$, the only $m+1$ squares are used. If we regard these squares as tracks in one square, these squares can be scanned by only one head simultaneously.

First of all, $M'$ writes $a_h$ on the first square of the first working tape of $M'$ if the first symbol of the input string $w$ is $a$, and copies the $i$-th symbol of the input string $w$ to the first square of the $i$-th working tape of $M'$ for $2 \leq i \leq |w|$. This operation completes at $|w|$ steps. The $j$-th square of the $i$-th working tape of $M$ ($1 \leq i \leq m, i \leq j < \infty$) corresponds to the $i + 1$-st square of the $j$-th working tape of $M'$. For example, if the situation of $M$ is as in Figure 3, then the corresponding one of $M'$ is as in Figure 4. A subscript $h$ in Figure 4 means that its square is the corresponding one of a head position in Figure 3.
On the Turing Machine with Linear-Time Transition Function

Let $M'$ simulate $2^{t+1}-2^t$ steps of $M$ at only one step for $0 \leq t \leq \varrho(|w|) - 1$.

Now, we consider the moves of $M$ from the $2^t$-th step to the $2^{t+1}$-th step. Each working tape of $M$ has scanned at most $2^t$ squares by time $2^t$. So, $M'$ scans all the squares of $2^t$ working tapes from the top simultaneously, $\delta'$ computes the following $2^{t+1}-2^t$ steps of $M$, and the consequence is written on $2^{t+1}$ working tapes. This is one step of $M'$. Since $M$ accepts or rejects $w$ within $2^{\varrho(|w|)}$ steps, $M'$ does it within $|w| + \varrho(|w|)$ steps. Since $f(x) \geq O(2^{cx})$ by assumption, $\varrho(x) = \log_2 f(x) \geq O(cx) = O(x)$. Therefore, $O(|w| + \varrho(|w|)) = O(\varrho(|w|)) = O(\log_2 f(|w|))$.

Next, the arguments of $\delta'$ in time $|w| + t$ are $\max(|w|, 2^t) + 2$, and the values of $\delta'$ are gotten at $2^{t+1}-2^t$ steps. So, $\frac{2^{t+1} - 2^t}{\max(|w|, 2^t) + 2} < \frac{2^{t+1}}{2^t} = 2$ (constant). On the other hand, at time $t \leq |w|$, the values of $\delta'$ are gotten at one step. Therefore, $\delta'$ is linear-time computable.

References


Department of Mathematics
College of Humanities & Sciences
Nihon University
Sakurajosui 3-25-40 Setagayaku
Tokyo 156 Japan