On the Radii of Starlikeness and Convexity for Certain Multivalent Functions

Teruo YAGUCHI
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We consider the class $A_p$ of functions $f(z)$ which are analytic in the unit disk and have the conditions $f(0)=f'(0)=\cdots=f^{(p-1)}(0)=0$ and $f^{(p)}(0)=p!$. The object of the present paper is to determine the radii of starlikeness and convexity of order $r$ for functions of certain subclass of the class $A_p$.

1. Introduction.

Let $p \in \mathbb{N} = \{1, 2, 3, \ldots\}$, $0 < \alpha < p$, $\beta > 0$ and $0 < r \leq 1$. Let $U_r$ denote the set $\{z : |z| < r\}$ and let $U$ denote the unit disk $U_1$. Next, let $A_p$ denote the class of functions $f(z)$ of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n,$$

which are analytic in the unit disk $U$. We denote $A_1$ by $A$.

A function $f(z)$ in the class $A_p$ is said to be $p$-valently starlike of order $\alpha$ in $U_r$ if and only if it satisfies

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in U_r).$$

We denote, by $S_p(\alpha)_r$, the subclass of the class $A_p$ consisting of all $p$-valently starlike functions of order $\alpha$ in $U_r$. Furthermore, a function $f(z)$ in the class $A_p$ is said to be $p$-valently convex of order $\alpha$ in $U_r$ if and only if it satisfies

$$\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in U_r).$$

Also, we denote by $K_p(\alpha)_r$ the subclass of the class $A_p$ consisting of all $p$-valently convex functions of order $\alpha$ in $U_r$.

Let $M$ be the class of functions $p(z)$ of the form:

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,$$

which are analytic in the unit disk $U$. A function $p(z)$ in the class $M$ is said to be a member of the class $M(\beta)$ if and only if it satisfies

$$|\text{arg} \ p(z)| < \frac{\pi \beta}{2} \quad (z \in U).$$

Finally, a function $f(z)$ in the class $A_p$ is said to be a member of the class $C_p(M_\beta, \alpha)$.
if and only if there is a function \( g(z) \in S'_p(\alpha) \) such that \( \frac{f(z)}{g(z)} \in M(\beta) \). A function \( f(z) \) in the class \( A_p \) is also said to be a member of the class \( C_p(M_p, \alpha) \) if and only if there is a function \( g(z) \in K_p(\alpha) \) such that \( \frac{f'(z)}{g'(z)} \in M(\beta) \).

In particular, whenever the numbers \( p, \alpha, \beta \) and \( r \) mentioned in technical terms \( S'_p(\alpha), K_p(\alpha), C'_p(M_p, \alpha) \) and \( C_p(M_p, \alpha) \) are equal to 1, 0, 1 and 1, respectively, these numbers are removed from the technical terms. For example,

\[
\begin{align*}
S'_p(\alpha) &= S'_p(\alpha), \quad K_p(\alpha) = K_p(\alpha), \quad C'_p(M_p, \alpha) = C'_p(M_p, \alpha), \\
S'_r(\alpha) &= S'_r(\alpha), \quad K(\alpha) = K(\alpha), \quad C(\alpha, \alpha) = C(\alpha, \alpha), \\
K &= K(0), \quad C = C(1).
\end{align*}
\]

A function \( f(z) \) in the classes \( S^*, K, K(\alpha), C \) and \( C(\alpha, \alpha) \) is said to be starlike, convex, convex of order \( \alpha \) in \( U \), close-to-convex and close-to-convex of type \( \alpha \), respectively.

2. Preliminaries.

Before getting our results, we here have to recall Lemma 2. A, 2. B and prove Lemma 2. 1.

**Lemma 2. A** (Nunokawa and Causey [1]). Let \( \beta > 0 \). If \( p(z) \in M(\beta) \), then

\[
|p'(z)| \leq \frac{2\beta}{1-|z|^2} \quad (z \in U).
\]

The following lemma is a generalization of Alexander’s Theorem.

**Lemma 2. B.** Let \( p \in N \) and \( 0 \leq \alpha < p \). Then we have

\[
f(z) \in K_p(\alpha) \quad \text{if and only if} \quad \frac{z}{p}f'(z) \in S'_p(\alpha).
\]

**Lemma 2. 1.** Let \( p \in N \) and \( 0 \leq \alpha < p \). If \( g(z) \in S'_p(\alpha) \), then

\[
\text{Re} \frac{zg'(z)}{g(z)} \geq (p-\alpha) \frac{1-|z|}{1+|z|} + \alpha \quad (z \in U).
\]

**Proof.** Defining the functions \( h(z) \) and \( h_0(z) \) by

\[
h(z) = \frac{zg'(z)}{p'g(z)} \quad \text{and} \quad h_0(z) = \frac{z}{p} \frac{1-\alpha}{1+\alpha} + \frac{\alpha}{p},
\]

respectively, we have

\[
h(z), h_0(z) \in M, \quad h(0) = h_0(0) \quad \text{and} \quad \text{Re} \ h(z) > \frac{\alpha}{p},
\]

because of \( g(z) \in S'_p(\alpha) \). Since the function \( h_0(z) \) is univalent in the unit disk \( U \) and maps the disk \( U \) onto \( \text{Re} \ w > \frac{\alpha}{p} \), we obtain

\[
|h_0^{-1}(h(z))| \leq |z| \quad (z \in U),
\]

by using Schwarz’s lemma. This inequality shows that the image of a point \( z \) in the unit disk \( U \) by the function \( h(z) \) has to be in the disk whose diameter end points are
3. The radius of starlikeness.

In this section, we have a theorem and four corollaries of the theorem.

**Theorem 3.1.** Let $p \in \mathbb{N}, j, l \in \{0, 1, 2, \ldots, p-1\}, 0 \leq \alpha < p-j, \beta > 0$ and $0 \leq \gamma < p-l$.

If a function $f(z)$ is in the class $A_p$ and satisfies

$$
(p-l)! z^l f^{(1)}(z) \in M(\beta) 
$$

for some $g(z) \in A_p$ with the condition

$$
\frac{(p-j)!}{p!} g^{(j)}(z) \in S_{\gamma}^\alpha, 
$$

then

$$
\frac{(p-l)!}{p!} f^{(1)}(z) \in S_{\gamma}^{p-j}(r),
$$

where

$$
\left\{ \begin{array}{ll}
\frac{p-j-\alpha+\beta-\sqrt{A}}{p+l-2j-2\alpha+\gamma} & \text{if } p+l-2j-2\alpha+\gamma > 0, \\
\frac{p-j-\alpha+\beta+\sqrt{A}}{p+l-2j-2\alpha+\gamma} & \text{if } p+l-2j-2\alpha+\gamma < 0, \\
\frac{p-j-\alpha}{p-j-\alpha+\beta} & \text{if } p+l-2j-2\alpha+\gamma = 0,
\end{array} \right.
$$

and

$$
A = (\alpha-\beta-\gamma+j-l)^2 + 2\beta(p-l-\gamma).
$$

The result is sharp for the function $f(z)$ defined by

$$
f(z) = \frac{z^p}{(1-z)^{2(p-j-\alpha)}} \left( \frac{1+z}{1-z} \right)^\beta (l=0),
$$

$$
f(z) = \left[ \int_0^1 \frac{p+1}{(1-\xi)^{2(p-j-\alpha)}} \left( \frac{1+\xi}{1-\xi} \right)^\beta d\xi \right] (l=1),
$$

and

$$
f(z) = \left[ \int_0^1 \int_0^1 \cdots \int_0^1 \frac{p+1}{(p-l)! (1-\zeta_1)^{2(p-j-\alpha)}} \left( \frac{1+\zeta_1}{1-\zeta_1} \right)^\beta d\zeta_1 \cdots d\zeta_{l-1} d\zeta_l \right] (l=2, 3, \ldots, p-1),
$$

at $z=-|z|$.

**Proof.** Defining the function $p(z)$ by

$$
p(z) = \frac{(p-l)!}{(p-j)!} z^l f^{(1)}(z) \left/ z^l g^{(j)}(z) \right.,
$$

we have $p(z) \in M(\beta)$. Since

$$
\frac{p^*(z)}{p(z)} = \frac{z^l f^{(l+1)}(z)}{f^{(1)}(z)} - \frac{z^l f^{(l+1)}(z)}{g^{(j)}(z)}
$$

we see, using Lemma 2. A, that

$$
\left| \frac{z^l f^{(l+1)}(z)}{f^{(1)}(z)} + l - j - \frac{z^l g^{(j+1)}(z)}{g^{(j)}(z)} \right| \leq \frac{2\beta|z|}{1-|z|^3} (z \in U).
$$
The function \( \frac{(p-j)!}{p!} g^{(p-j)}(z) \in S_{p-j}(\alpha) \) satisfies

\[
\Re \left( \frac{z g^{(p-j+1)}(z)}{g^{(p)}(z)} \right) \geq \Re \left( \frac{z g^{(p-j+1)}(z)}{g^{(p)}(z)} \right) + j - l - \frac{2\beta|z|^2}{1 - |z|^2} \geq (p-j-a) \frac{1 - |z|}{1 + |z|} + \alpha + j - l - \frac{2\beta|z|^2}{1 - |z|^2} > \gamma
\]
in \( U_r \), which is equivalent to the inequality

\[
(p + l + \gamma - 2j - 2a)|z|^2 - 2(p - j - a + \beta)|z| + p - \gamma - l > 0
\]
in \( U_r \), where \( r \) is defined by (3.4).

We here obtain many corollaries. Putting \( l = j \) in Theorem 3.1, we have Corollary 3.1.

**Corollary 3.1.** Let \( p \in \mathbb{N}, j \in \{0, 1, 2, \ldots, p-1\}, 0 \leq \alpha < p-j, \beta > 0 \) and \( 0 \leq \gamma < p-j \). If a function \( f(z) \) is in the class \( A_p \) and satisfies

\[
\frac{(p-j)!}{p!} f^{(p-j)}(z) \in S_{p-j}(\gamma)r,
\]
where

\[
\left\{ \begin{array}{lr}
r = \frac{p-j-a+\beta-\sqrt{B}}{p-j-2a+\gamma} & \text{if } p-j-2a+\gamma > 0, \\
r = \frac{p-j-a+\beta+\sqrt{B}}{p-j-2a+\gamma} & \text{if } p-j-2a+\gamma < 0, \\
r = \frac{p-j-a}{p-j-\alpha+\beta} & \text{if } p-j-2a+\gamma = 0,
\end{array} \right.
\]

and

\[
B = (\alpha - \beta - \gamma)^2 + 2\beta(p-j-\gamma).
\]
The result is sharp for the function \( f(z) \) defined by

\[
f(z) = \frac{z^p}{(1-z)^{2(p-j-\gamma)}} \left( \frac{1+z}{1-z} \right)^j \quad (j = 0),
\]

\[
f(z) = \int_0^z \left( \frac{\zeta^{p-1}}{(1-\zeta)^{2(p-j-\gamma)}} \left( \frac{1+\zeta}{1-\zeta} \right)^j \right) d\zeta \quad (j = 1),
\]

and

\[
f(z) = \int_0^{z_1} \cdots \int_0^{z_{j-1}} \frac{\zeta^j}{(p-j)! (1-\zeta_1)^{2(p-j-\gamma)}} \left( \frac{1+\zeta_1}{1-\zeta_1} \right)^j d\zeta_1 \cdots d\zeta_{j-1} d\zeta_j \quad (j = 2, 3, \ldots, p-1),
\]
at \( z = -|z| \).

Putting \( j = 0 \) in Corollary 3.1, we have Corollary 3.2.
Corollary 3.2. Let \( p \in \mathbb{N}, 0 \leq \alpha < p, \beta > 0 \) and \( 0 \leq \gamma < p \). If \( f(z) \in C_p(M_p, \alpha) \), then
\[
f(z) \in S_p(\gamma)_r, \]
where
\[
\begin{align*}
r &= \frac{p - \alpha + \beta - \sqrt{C}}{p - 2\alpha + \gamma} \quad \text{if} \quad p - 2\alpha + \gamma > 0, \\
r &= \frac{p - \alpha + \beta + \sqrt{C}}{p - 2\alpha + \gamma} \quad \text{if} \quad p - 2\alpha + \gamma < 0, \\
r &= \frac{p - \alpha}{p - \alpha + \beta} \quad \text{if} \quad p - 2\alpha + \gamma = 0,
\end{align*}
\]
and
\[
C = (\alpha - \beta - \gamma)^2 + 2\beta(p - \gamma).
\]
The result is sharp for the function \( f(z) \) defined by (3.19) at \( z = -|z| \).

Putting \( p = 1 \) in Corollary 3.2, we have Corollary 3.3.

Corollary 3.3. Let \( 0 \leq \alpha < 1, \beta > 0 \) and \( 0 \leq \gamma < 1 \). If \( f(z) \in C^*(M_p, \alpha) \), then \( f(z) \in S^*(\gamma)_r \), where
\[
\begin{align*}
r &= \frac{1 - \alpha + \beta - \sqrt{D}}{1 - 2\alpha + \gamma} \quad \text{if} \quad 1 - 2\alpha + \gamma > 0, \\
r &= \frac{1 - \alpha + \beta + \sqrt{D}}{1 - 2\alpha + \gamma} \quad \text{if} \quad 1 - 2\alpha + \gamma < 0, \\
r &= \frac{1 - \alpha}{1 - \alpha + \beta} \quad \text{if} \quad 1 - 2\alpha + \gamma = 0,
\end{align*}
\]
and
\[
D = (\alpha - \beta - \gamma)^2 + 2\beta(1 - \gamma).
\]
The result is sharp for the function \( f(z) \) defined by
\[
f(z) = \frac{z}{(1 - z)^{2(1 - \alpha)}} \left( \frac{1 + z}{1 - z} \right)^{\beta} \]
at \( z = -|z| \).

Remark 3.1. Taking \( \alpha = 0 \) in Corollary 3.3, we have the corresponding result due to Yaguchi, Obradović, Nunokawa and Owa [3].

Putting \( \gamma = \alpha \) in Corollary 3.3, we have the following corollary.

Corollary 3.4. Let \( 0 \leq \alpha < 1 \) and \( \beta > 0 \). If \( f(z) \in C^*(M_p, \alpha) \), then \( f(z) \in S^*(\alpha)_r \), where
\[
r = 1 - \frac{\sqrt{E} - \beta}{1 - \alpha}
\]
and
\[
E = \beta^2 + 2\beta(1 - \alpha).
\]
The result is sharp for the function \( f(z) \) defined by (3.26) at \( z = -|z| \).

Remark 3.2. Taking \( \alpha = 0 \) and \( \beta = 2 \) in Corollary 3.4, we have the corresponding result due to Yaguchi and Nunokawa [2].
4. The radius of convexity.

We obtain the following result with the aid of Theorem 3.1 and Lemma 2. B.

**Theorem 4.1.** Let \( p, j, l, \alpha, \beta \) and \( \gamma \) have the same conditions as in Theorem 3.1. If a function \( f(z) \) is in the class \( A_p \) and satisfies

\[
\frac{(p-l-1)!}{(p-j-1)!} \frac{z^l f^{(l+1)}(z)}{z^j g^{(j+1)}(z)} \in M(\beta)
\]

for some \( g(z) \in A_p \) with the condition

\[
\frac{(p-j-1)!}{p!} z^j g^{(j+1)}(z) \in K_{p-1}(\gamma),
\]

then

\[
\frac{(p-l-1)!}{p!} f^{(l)}(z) \in K_{p-l}(\gamma),
\]

where \( r \) is given by (3.4). The result is sharp for the function \( f(z) \) defined by (3.7) \((l=0)\), and the function

\[
f(z) = \int_0^1 \cdots \int_0^{\xi_l} \frac{p!}{(p-l-1)!} \frac{z^l}{(1-\xi_0)^{(p-j-\gamma)}} \frac{1}{(1-\xi_0)^{\beta}} d\xi_0 \cdots d\xi_{l-1} d\xi_l,
\]

\((l=1, 2, \ldots, p-1)\)
at \( z = -|z| \).

**Proof.** Let the functions \( F(z) \) and \( G(z) \) in the class \( A_p \) be defined by

\[
F^{(l)}(z) = \frac{x}{p-l} f^{(l+1)}(z)
\]

and

\[
G^{(j)}(z) = \frac{x}{p-j} g^{(j+1)}(z),
\]

respectively. Then we have, with the aid of Lemma 2. B, (4.1) and (4.2),

\[
\frac{(p-j-1)!}{p!} G^{(j)}(z) \in S^*_p-j(\alpha),
\]

and

\[
\frac{(p-l-1)!}{(p-j-1)!} z^l F^{(l)}(z) = \frac{(p-l-1)!}{(p-j-1)!} z^l f^{(l+1)}(z) \in M(\beta).
\]

By Theorem 3.1, we also obtain

\[
\frac{(p-l-1)!}{p!} F^{(l)}(z) \in S^*_p-l(\gamma),
\]

where \( r \) is given by (3.4). Using Lemma 2. B again, we have that \( \frac{(p-l-1)!}{p!} f^{(l)}(z) \) is \((p-l)\)-valently convex of order \( \gamma \) in \( U_r \).

Putting \( l = j \) in Theorem 4.1, we have Corollary 4.1.

**Corollary 4.1.** Let \( p, j, \alpha, \beta \) and \( \gamma \) have the same conditions as in Corollary 3.1. If a function \( f(z) \) is in the class \( A_p \) and satisfies

\[
\frac{f^{(j+1)}(z)}{g^{(j+1)}(z)} \in M(\beta)
\]

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for some \( g(z) \in \mathcal{A}_p \) with the condition (4.2), then
\[
(4.11) \quad \frac{(p-j+1)!}{p!} f^{(j)}(z) \in K_{p-j}(\gamma)_r,
\]
where \( r \) is given by (3.17). The result is sharp for the function \( f(z) \) defined by
\[
(4.12) \quad f(z) = \int_0^z \frac{\xi^{p-1}}{(1-\xi)^2} \left( \frac{1}{1-\xi} \right)^j d\xi, \quad (j=0),
\]
\[
(4.13) \quad f(z) = \int_0^z \cdots \int_0^z \frac{p!}{(p-j)!} \left( \frac{1}{1-\xi} \right)^j d\xi_0 \cdots d\xi_{j-1} d\xi_j,
\]
\( (j=1, 2, \ldots, p-1) \)
at \( z = -|z| \).

Putting \( j = 0 \) in Corollary 4.1, we have Corollary 4.2.

**Corollary 4.2.** Let \( p, \alpha, \beta \) and \( \gamma \) have the same conditions as in Corollary 3.2. If \( f(z) \in \mathcal{C}_p(M_\beta, \alpha) \), then \( f(z) \in \mathcal{K}_\alpha(\gamma)_r \), where \( r \) is given by (3.22). The result is sharp for the function \( f(z) \) defined by (4.12) at \( z = -|z| \).

Putting \( p = 1 \) in Corollary 4.2, we have Corollary 4.3.

**Corollary 4.3.** Let \( \alpha, \beta \) and \( \gamma \) have the same conditions as in Corollary 3.3. If \( f(z) \in \mathcal{C}(M_\alpha, \alpha) \), then \( f(z) \in \mathcal{K}(\gamma)_r \), where \( r \) is given by (3.24). The result is sharp for the function \( f(z) \) defined by
\[
(4.14) \quad f(z) = \int_0^z \frac{1}{(1-\xi)^2} \left( \frac{1}{1-\xi} \right)^j d\xi,
\]
at \( z = -|z| \).

**Remark 4.1.** Taking \( \alpha = 0 \) in Corollary 4.3, we have the corresponding result due to Yaguchi, Obradović, Nunokawa and Owa [3].

Putting \( \beta = 1 \) in Corollary 4.3, we have Corollary 4.4.

**Corollary 4.4.** Let \( 0 \leq \alpha < 1 \) and \( 0 \leq \gamma < 1 \). If a function \( f(z) \) in the class \( A \) is close-to-convex of type \( \alpha \), then the function \( f(z) \) is convex of order \( \gamma \) in \( U_r \), where
\[
(4.15) \quad r = \begin{cases} 
\frac{2-\alpha - \sqrt{F}}{1-2\alpha + \gamma} & \text{if } 1-2\alpha + \gamma > 0, \\
\frac{2-\alpha + \sqrt{F}}{1-2\alpha + \gamma} & \text{if } 1-2\alpha + \gamma < 0, \\
\frac{1-\alpha}{2-\alpha} & \text{if } 1-2\alpha + \gamma = 0,
\end{cases}
\]
and
\[
(4.16) \quad F = (1-\alpha + \gamma)^2 + 2(1-\gamma).
\]
The result is sharp for the function \( f(z) \) defined by
\[
(4.17) \quad f(z) = \frac{1}{(2\alpha-1)(1-\alpha)} \left( \frac{\alpha - (1-\alpha)z}{(1-\alpha)^2 - 1} - \alpha \right) \left( \alpha = \frac{1}{2} \right),
\]
\[
(4.18) \quad f(z) = \frac{2\alpha}{1-z} + \log (1-z) \quad \left( \alpha = \frac{1}{2} \right),
\]
at \( z = -|z| \).
Putting \( a = 0 \) in Corollary 4.4, we have Corollary 4.5.

**Corollary 4.5.** Let \( 0 \leq \gamma < 1 \). If a function \( f(z) \) in the class \( A \) is close-to-convex, then the function \( f(z) \) is convex of order \( \gamma \) in \( U_r \), where \( r = \frac{2 - \sqrt{3 + \gamma^2}}{1 + \gamma} \). The result is sharp for the Koebe function \( f(z) = \frac{z}{(1-z)^2} \) at \( z = -|z| \).

**References**


Department of Mathematics
College of Humanities and Sciences
Nihon University
3-25-40 Sakurajousui, Setagaya,
Tokyo, 156 JAPAN