Hyperbolic Sets and Axiom A for Endomorphisms

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Abstract
The properties of hyperbolic sets of endomorphisms and nonwandering set of endomorphisms are studied by F. Przytycki [P1], [P2]. This paper gives improved detailed proof to these properties.

§ 1 Introduction
An endomorphism \( f \) on a compact manifold \( M \) is a map of \( M \) into itself of class \( C^r \), \( r \geq 1 \). For the study of the global orbit structure of diffeomorphisms on compact manifolds, many powerful techniques and results have been exhibited. After, this the corresponding studies of endomorphisms have been exhibited [SH], [M], [MP], [P1], [P2], [Ikd], [AK], and [Ikg].

Let \( M \) be a compact manifold of dimension \( m \) with a Riemannian metric. We say a mapping \( f \) of \( M \) into itself an endomorphism. Let \( f \) be an endomorphism of class \( C^r \), \( r \geq 1 \). Let \( A \) be a closed subset with \( f(A) = A \). F. Przytycki gives an improved definition of hyperbolic sets for endomorphisms [P1] (see Definition 1.1).

1.1 Definition: (F. Przytycki, [P1]) An invariant set \( A \) is called a hyperbolic set of \( f \) if there exist real constants \( C>0, 0<\mu<1 \) such that for every \( f \)-orbit \( (x_n) \), \( n \in \mathbb{Z} \), of points in \( A \) and for every integer \( i \) we have:

\[
T_xM = E^s(x_i) \oplus E^u(x_i)
\]

\[
Tf(E^s(x_i)) = E^s(x_{i-1}), \quad ||Tf^i_s(v)|| \leq C\mu^n ||v|| \quad \text{for} \quad v \in E^s(x_i)
\]

\[
Tf(E^u(x_i)) = E^u(x_{i-1}), \quad ||Tf^i_u(v)|| \geq C^{-1}\mu^{-n} ||v|| \quad \text{for} \quad v \in E^u(x_i),
\]

\[
T_xM = E^s(x_i) \oplus E^u(x_i)
\]

\[
Tf(E^s(x_i)) = E^s(x_{i-1}), \quad ||Tf^i_s(v)|| \leq C\mu^n ||v|| \quad \text{for} \quad v \in E^s(x_i)
\]

\[
Tf(E^u(x_i)) = E^u(x_{i-1}), \quad ||Tf^i_u(v)|| \geq C^{-1}\mu^{-n} ||v|| \quad \text{for} \quad v \in E^u(x_i),
\]

\[
n = 0, 1, 2, \ldots.
\]

We say \( A \) has skewness \( \mu \).

If \( f \) is a diffeomorphism then Definition 1.1 implies that \( A \) is a hyperbolic set of \( f \) in the sense of [Sm]. In fact, the orbit of \( x \in A \) is unique and \( T_xM = E^s(x) \oplus E^u(x) \) in Definition 1.1 is continuous for the variable \( x \in M \) by [P1, Theorem 1.10].

The purpose of this paper is to show the fundamental properties of hyperbolic sets for endomorphisms and to give the proofs of these properties. In this paper, we give proofs to some properties, which are mentioned without proof by F. Przytycki in [P1] and [P2].

In § 1 the properties of hyperbolic sets are provided. The main results in this section are Proposition 2.6 and Proposition 2.7 (or Corollary 2.8). In § 3 we provide the funda-
mental property of Axiom A endomorphisms.

§ 2 Properties of Hyperbolic Sets

The purpose of this section is Proposition 2.6, Proposition 2.7 and Corollary 2.8. Throughout in this section let $U$ be an open subset of $M$, $f: U \rightarrow M$ be an endomorphism of class $C^r$, $1 \leq r < \infty$, and $A$ be a hyperbolic set of $f$.

Let $HM := \bigoplus_{i=-\infty}^{\infty} M_i$, where $M_i$ is the copy of $M$. Define a metric $d$ on $M$ as follows:

$$d(x, y) := \sum_{i=-\infty}^{\infty} \frac{1}{2|\lambda|} d(x_i, y_i), \quad x, y \in M, \quad x_i, y_i \in M_i,$$

where $d$ is a metric of $M$. Then, $(HM, d)$ is a compact metric space. For a continuous surjection $f: M \rightarrow M$, the set

$$M' := \{x = (x_i) \in HM | f(x_i) = x_{i+1}\}$$

is a closed subset of $HM$. If $A$ is a closed subset of $M$ such that $f(A) = A$, the set $\tilde{A} := A^{\prime \prime}$ is a closed subset of $M'$. Hence, $\tilde{A}$ is compact.

In our case, $A$ is a hyperbolic set of an endomorphism $f : M \rightarrow M$. Let $\tilde{x} = (x_i)$ be an orbit in $A$ and

$$T_{\tilde{x}} M = E^h \tilde{x}(x) \oplus E^s \tilde{x}(x) \quad (2.1)$$

be the splitting as Definition 1.1 at $x := x_0$ associated to $\tilde{x}$. By [P1, Theorem 1.10] and its proof, this splitting is uniquely associated to $\tilde{x}$ and the map $\tilde{x} \rightarrow E^h \tilde{x}, E^s \tilde{x}$ are continuous as the mapping from $\tilde{A}$ to the Grassmann bundles on $A$.

2.1 Lemma. ([P1, Proposition 1.4]) There exists a smooth Riemannian metric $\langle \cdot, \cdot \rangle_A$ on $M$ adapted to $A$ such that for some $\lambda : 0 < \lambda < 1$ and for every $f$-orbit $\tilde{x} = (x_i)$ in $A$, putting $x = x_0$,

$$|\langle Tf(x) \rangle|_A \leq \lambda |\langle v \rangle|_A \quad \text{for} \quad v \in E^h \tilde{x}(x) \quad (2.2)$$

$$|\langle Tf(x)v \rangle|_A \leq \lambda^{-1} |\langle v \rangle|_A \quad \text{for} \quad v \in E^s \tilde{x}(x) \quad (2.3)$$

Proof: Let $A$ be a hyperbolic set satisfying (1.1), (1.2), and let $\langle \cdot, \cdot \rangle$ be the given inner product of the vectors $u, v \in T_{\tilde{x}} M$, $x \in A$. We define a new inner product $\langle \cdot, \cdot \rangle_A$ on $T_{\tilde{x}} M$ as follows:

$$\langle u, v \rangle_A := \sum_{j=0}^{N-1} \langle Tf^j(u), Tf^j(v) \rangle, \quad u, v \in T_{\tilde{x}} M, \quad x \in A,$$

where $N$ is so large that $C^s < 1$. Let $|| \cdot ||_A$ be the norm on $T_{\tilde{x}} M$, $x \in A$, induced from $\langle \cdot, \cdot \rangle_A$. For an $f$-orbit $\tilde{x} = (x_i)$, $x_0 = x$, let $u \in E^s \tilde{x}(x)$ be a non-zero vector. Using

$$\sum_{j=0}^{N-1} |\langle Tf^j u \rangle|^2 = \sum_{j=0}^{N-1} C^s |u|_A^2 |\langle u \rangle|^2 \leq NC^s |u|_A^2,$$

we have

$$|\langle Tf u \rangle|_A^2 = \sum_{j=0}^{N-1} |\langle Tf^{j+1} u \rangle|^2$$

$$= |\langle u \rangle|_A^2 - |\langle u \rangle|_A^2 + |\langle Tf^{j+1} u \rangle|^2$$

$$\leq |\langle u \rangle|_A^2 - (1 - (C^s)^2) |\langle u \rangle|_A^2$$

$$= 70$$
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\[ \leq \left( 1 - \left( 1 - C_p \right)^2 \right) \frac{||u||}{||u||}. \]

Putting \( K_s = (1 - (1 - C_p)^2)^{-1/2} \), we have \( 0 < K_s < 1 \) and \( ||T f u||_A \leq K_s ||u||_A \).

Next, let \( v \in E^u v(x) \). Then, we have

\[ \frac{||T f u||^2}{||v||^2} = \frac{||v||^2 - ||v||^2 + ||T f u||^2}{||v||^2} \]

\[ = 1 + \frac{||T f u||^2 - ||v||^2}{||v||^2} \]

\[ \geq 1 + \frac{(C_p^2 - 1)||v||^2}{||v||^2 + \cdots + ||T f u||^2} = K(v). \]

We may assume that \( v \) belongs to the unit sphere \( S^{n-1}(x) \subseteq E^u v(x) \). Consider the sets

\[ E^u = \{ v \in E^u v(x) | \exists x \in A \}, \]

\[ S^u = \{ v \in S^{n-1}(x) | \exists x \in A \}. \]

Let \( \tau : TM \to M \) be the tangent bundle, and let \( \pi_0 : A \to A \) be the projection defined by \( \pi_0(\hat{x}) = x \). Since \( E^u v(x) \) depends on \( \hat{x} \) continuously, it follows that \( E^u \) is a subbundle of the induced bundle \( \pi_0 \tau \).

\( S^u \) is compact since \( S^u \) is a subbundle of \( E^u \) and \( A \) is compact. The number \( K(v) > 1 \) depends continuously on \( v \). Hence, there is \( K > 1 \) such that \( K(v) > (K_s)^2 \) for every \( v \in E^u v(x) \) and every \( \hat{x} \in \hat{A} \).

Therefore, \( \lambda = \max \{ K_s, (K_s)^{-1} \} \) satisfies (2.2) and (2.3). \( \square \)

Let \( \pi : A \to A \) be the Projection defined by \( \pi(\hat{x}) = x_t \), where \( \hat{x} = (x_t) \). Define \( \tilde{f} : A \to A \) by \( \pi_0 \tilde{f}(\hat{x}) = x_{t+1} \). \( \tilde{f} \) is a homeomorphism. Then, \( (\hat{A}, \tilde{f}) \) is considered as the inverse limit \( \lim_{\to} (A, f) \).

2.2 Lemma. Let \( \sigma : X \to X \) be a homeomorphism on a topological space and \( f : A \to A \) be a continuous mapping. Let \( \tilde{f} : A \to A \) be the inverse limit of \( (A, f) \). Then we have the following:

(i) For any continuous map \( h : X \to A \) satisfying \( h \sigma = f h \), there exists uniquely a continuous map \( \tilde{h} : X \to A \) such that \( \pi_0 \tilde{h} = h \), consequently \( h \sigma = \tilde{f} h \).

(ii) If \( h \) is surjective, then \( \tilde{h} \) is surjective

[Diagram]

Proof: (i) For \( x \in X \), define \( \tilde{h}(x) = \hat{y} = (y_t) \) by \( y_t := h \sigma^t(x) \). Since \( \pi_0 \tilde{h}(\hat{x}) = \pi_0 (\hat{y}) = y_0 = h(x) \), we have \( \pi_0 \tilde{h} = h \). Then, we have

\[ \pi_0 \tilde{f} \tilde{h}(x) = \pi_0 \tilde{h} \tilde{f}(x) = h \sigma^{t+1}(x) \]

\[ = h \sigma^t(\sigma x) = \pi_0 \tilde{h} \sigma(x) \]

for every \( t \in \mathbb{Z} \), i.e. \( \tilde{f} \tilde{h} = \tilde{h} \sigma \). For showing uniqueness, let \( h : X \to A \) satisfy \( \pi_0 h = h \). Since \( \tilde{f} \tilde{h}(x) = h \sigma^t(x) \) as verified above, it follows that

\[ \tilde{h} = \tilde{f} h \]

(21)
Therefore \( h = \tilde{h} \). Clearly, \( h \) is continuous.

(ii) Let \( (p_i) \in \mathcal{A} \) be given. For any \( i \in \mathbb{Z} \) there is \( x_i \in X \) with \( h(x_i) = p_i \in A \), by the assumption. We have

\[
\pi_i h(x_i) = h \sigma^i(x_i) = f^n h(x_i) = p_{i+n}, \quad n \geq 0.
\]

It follows that, putting \( x^j := \sigma^j(x_{-j}), \ i \geq 0, \)

\[
\lim_{j=\infty} h(x^j) = (p_i).
\]

Since \( X \) is compact, it follows that some subsequence of \( \{x^j\}_{j=0} \) converges to an element \( x \in X \). By the continuity of \( h \), we have \( h(x) = (p_i) \). \( \blacksquare \)

Let \( C^r(L, M) \) be the space of all \( C^r \) bounded maps from \( L \) into \( M \) with uniform \( C^r \) topology, \( 0 \leq r < \infty \). Here, \( L \) and \( M \) are finite dimensional smooth manifolds without boundary, however, if \( r = 0 \), \( L \) is assumed to be of class \( C^0 \). In this paper we do not use the structure as a manifold but a local structure of \( C^r(L, M) \). So, we need only the manifold chart of \( C^r(L, M) \) defined as follows.

Let \( \tau : TM \to M \) be the tangent bundle of \( M \). For a given map \( h \in C^r(L, M) \), consider the set

\[
h^*\tau := \{(x, v) \mid x \in L, v \in T_h(x)M\}
\]
as a subspace of \( L \times TM \). We have a vector bundle \( \tau' : h^*\tau \to L \) defined by \( \tau'(x, v) := x \). Let \( \Gamma^r(h^*\tau) \) be the space of all \( C^r \) sections of \( \tau' \). Let \( N_h \) be a small neighbourhood of \( h \) in \( C^r(L, M) \). We define a homeomorphism onto the image \( \Gamma(h^*\tau) \) as follows. For \( h' \in N_h \) we define \( \xi := \varphi(h') \) by

\[
\xi(x) := \exp_{x^i} h'(x), \quad x \in L.
\]

We call \( (N_h, \varphi) \) a manifold chart with center \( y \).

The following lemma is a direct consequence.

2.3 Lemma. If \( f : L \to M \) is a \( C^r \) map, then the map \( f^* : C^r(M, N) \to C^r(L, N) \), defined by \( g \to gf \), is \( C^r \). Here, \( L \) is assumed to be merely topological spaces if \( r = 0 \).

2.4 Lemma. If \( L \) is compact, then the map

\[
\text{comp} : C^r(L, M) \times C^{r+s}(M, N) \to C^r(L, N)
\]
defined by \( (h, g) \to gh \) is \( C^r \). Here, \( L, M \) are assumed to be topological spaces if \( r = 0, r + s = 0 \), respectively.

Proof: By M.C. Irwin [Ir, Theorem B.15] our assertion is proved when \( M \) and \( N \) are open subsets of Euclidean spaces. We reduce our situation to this case. Let regard \( M \) and \( N \) as submanifolds of \( \mathbb{R}^m \) and \( \mathbb{R}^n \), respectively.

The inclusion map \( i_N : M \to \mathbb{R}^m \) induces the smooth inclusion map

\[
(i_M)_* : C^r(L, M) \to C^r(L, \mathbb{R}^m).
\]

In fact let \( \varphi : N_h \to \Gamma^r(h^*\tau) \) be a chart with center \( h \). The diagram

\[ (22) \]
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\[
\begin{align*}
TM & \longrightarrow \mathbb{R}^m \times \mathbb{R}^m \\
\tau & \hookrightarrow \Gamma
\end{align*}
\]

induces the bundle inclusion \( h^*\tau \subset (i_M h)^*\tau \) and the inclusion \( \Gamma^r((i_M h)^*\tau) \subset \Gamma^r((i_M h)^*\pi) \).

Hence, we have the natural inclusion \( \tilde{\mathcal{N}} \subset \tilde{\mathcal{N}}_{(i_M h)^*\pi} \), where \( \tilde{\mathcal{N}}_{(i_M h)^*\pi} \rightarrow \Gamma((i_M h)^*\pi) \) is the chart of \( C^r(L, \mathbb{R}^m) \).

Let \( \tilde{M} \) be an open tubular neighbourhood of \( M \) in \( \mathbb{R}^m \) and \( p_m : \tilde{M} \rightarrow M \) the projection of the normal disk bundle. By Lemma 2.3, \( (p_m)^* : C^{r+\varepsilon}(M, N) \rightarrow C^{r+\varepsilon}(\tilde{M}, N) \) is \( C^{r+\varepsilon} \).

The inclusion map \( i_N : N \rightarrow \mathbb{R}^n \) induces as above the smooth inclusion maps \((i_N)_* : C^{r+\varepsilon}(\tilde{M}, N) \rightarrow C^{r+\varepsilon}(M, \mathbb{R}^n) \) and \((i_N)_* : C^r(L, N) \rightarrow C^r(L, \mathbb{R}^n) \). Therefore, the composition

\[
(p_m)^\sharp := (i_N)_*(p_m)^* : C^{r+\varepsilon}(M, N) \rightarrow C^{r+\varepsilon}(\tilde{M}, \mathbb{R}^n)
\]

is a smooth injection.

Hence, we have the commutative diagram

\[
\begin{array}{ccc}
C^r(L, M) \times C^{r+\varepsilon}(M, N) & \overset{\text{comp}}{\longrightarrow} & C^r(L, N) \\
\downarrow (i_M)_* \times (p_m)^\sharp & & \downarrow (i_N)_* \\
C^r(L, \tilde{M}) \times C^{r+\varepsilon}(\tilde{M}, \mathbb{R}^n) & \overset{\text{comp}}{\longrightarrow} & C^r(L, \mathbb{R}^n),
\end{array}
\]

where \( \text{comp} ((i_M)_* \times (p_m)^\sharp) \) is \( C^r \) by [Ir, Theorem B.15] and \((i_N)_* \) is a smooth inclusion. Therefore, we have that \( \text{comp} \) is \( C^r \).

Let \( f : U \rightarrow M \) be a \( C^r \) map from an open subset \( U \) of \( M \). Let \( \sigma : X \rightarrow X \) be a homeomorphism on a compact topological space \( X \). Define a map

\[
\mathcal{H}_f : C^0(X, U) \rightarrow C^0(X, M)
\]

by \( \mathcal{H}_f(h) := fh\sigma^{-1} \) for \( h \in C^0(X, U) \).

2.5 Proposition. (i) \( \mathcal{H}_f \) is a \( C^r \) map. Furthermore, the map \( C^r(U, M) \rightarrow C^r(C^0(X, U), C^0(X, M)) \), defined by \( f \rightarrow \mathcal{H}_f \), is continuous.

(ii) If \( h \in C^0(X, U) \) is a fixed point of \( \mathcal{H}_f \), then \( \Lambda := h(X) \) is an invariant set of \( f \). Furthermore, \( \Lambda \) is a hyperbolic set of \( f \) with skewness \( \eta \), if and only if \( h \) is a hyperbolic fixed point of \( \mathcal{H}_f \) with skewness \( \eta \).

Proof: (i) By Lemma 2.3, \((\sigma^{-1})^* : C^0(X, U) \rightarrow C^0(X, U)\) is \( C^\infty \). By Lemma 2.4, \( \text{comp} : C^0(X, U) \times C^r(U, M) \rightarrow C^0(X, M) \) is \( C^r \). Hence, the map \( f_* : C^0(X, U) \rightarrow C^0(X, M) \), defined by \( f_* : f \mapsto fh \) is \( C^r \), and the map \( C^r(U, M) \rightarrow C^r(C^0(X, U), C^0(X, M)) \) defined by \( f \mapsto f_* \) is \( C^r \). Therefore, \( \mathcal{H}_f = f_*((\sigma^{-1})^*) \) is \( C^r \). Since we can verify easily that Lemma 2.3 holds in the case of \( L, M, N \) being Banach manifolds, it follows that the map

\[
C^r(C^0(X, U), C^0(X, M)) \rightarrow C^r(C^0(X, U)), C^0(X, M))
\]

defined by \( f_* \mapsto f_*((\sigma^{-1})^*) \) is \( C^\infty \). Therefore, the correspondence \( f \mapsto \mathcal{H}_f = f_*((\sigma^{-1})^*) \) is continuous.

(ii) Since the first statement is trivial, we need only proof the second statement. Suppose that \( \Lambda \) is a hyperbolic set with skewness \( \eta \). The tangent space of \( C^0(X, U) \) at \( h \)
is $\Gamma^0(h^*\tau)$. The derivative of $\mathcal{H}_f$ at $h$ is given as follows, by [Ir, Theorem B. 10]:

$$(T_h\mathcal{H}_f)(\xi) = (T_f)\xi \sigma^{-1}, \quad \xi \in \Gamma^0(h^*\tau).$$

(2.4)

We show that $TH_f : \Gamma^0(h^*\tau) \to \Gamma^0(h^*\tau)$ is a linear homeomorphism as follows. Since $T_{pf}$ is nondegenerate at any point $p \in h(X) = A$, it follows that $T\mathcal{H}_f$ is injective. To show the surjectivity, let $\xi$ be any element of $\Gamma^0(h^*\tau)$. For any $x \in X$ there exists $\tilde{\xi}'(X) \in T_hM$ uniquely such that $Tf(\tilde{\xi}'(x)) = \xi(fh(x)) = \xi(\sigma(x))$ since $T_h(x)f$ is nondegenerate. And the correspondence $x \mapsto \tilde{\xi}'(x)$ is continuous by the same reason. Hence, $T\mathcal{H}_f$ is a bijection. Since $\tilde{\xi}'$ depends on $\xi$ continuously, it follows the continuity of $(T\mathcal{H}_f)^{-1}$.

Let $\tilde{h} : X \to A$ be the map obtained by Lemma 2.2 from $h$. Since $\tilde{h}(x)$ is an $f$-orbit in $A$ for $x \in X$, as (2.1) we have the splitting

$$T_{h(x))}M = E_{\tilde{h}(x)}(h(x)) \oplus E_{\tilde{h}(x)}(h(x)).$$

For the simplicity of notations, put

$$E^s(x) := E_{\tilde{h}(x)}(h(x)), \quad E^u(x) := E_{\tilde{h}(x)}(h(x)).$$

By [Pl, Theorem 1.10] and its proof, this splitting is uniquely associated to $x$ and varies continuously when $x$ varies since $h$ is continuous. Therefore, we have a splitting of the bundle $\tau' : h^*\tau \to X$

$$h^*\tau = \mathcal{E}^s \oplus \mathcal{E}^u,$$

$$\mathcal{E}^s := \{(x, v) \in X \times TM \mid v \in E^s(x)\} \subset h^*\tau,$$

$$\mathcal{E}^u := \{(x, v) \in X \times TM \mid v \in E^u(x)\} \subset h^*\tau,$$

$$\tau' : \mathcal{E}^s \to X, \quad \tau'^{u} : \mathcal{E}^u \to X.$$

It follows that $T_h\mathcal{H}_f$ has an invariant splitting

$$\Gamma^0(h^*\tau) = \Gamma^0(\mathcal{E}^s) \oplus \Gamma^0(\mathcal{E}^u).$$

(2.5)

For $\xi \in \Gamma^0(h^*\tau)$, let $||\xi|| := \sup \{|\xi x| \mid x \in X\}$, where $||\xi x||$ is the norm induced from the Riemannian metric on $M$. For given $\xi \in \Gamma^0(E^s)$, and $n > 0$ and there is $x \in X$ satisfying

$$||((T_h\mathcal{H}_f)^n \xi) x|| = ||Tf^n(\xi) x|| = C_{\eta^n}||\xi \sigma^{-n} x|| \leq C_{\eta^n}||\xi||.$$

Similarly, for $\xi \in \Gamma^0(E^u)$ and $n > 0$ we have

$$||\xi|| = ||\xi x|| \leq C_{\eta^n}||Tf^n(\xi) x|| = C_{\eta^n}||((T_h\mathcal{H}_f)^n \xi) \sigma^{-n} x|| \leq C_{\eta^n}||T_h\mathcal{H}_f\xi||.$$  

(2.6)

Therefore, $h$ is a hyperbolic fixed point of $H_f$ with skewness $\eta$.

Conversely, assume that $h$ is a hyperbolic fixed point of $H_f$ with skewness $\eta$. Then, we have a $T_h\mathcal{H}_f$-invariant splitting with skewness $\eta$.

$$\Gamma^0(h^*\tau) = (\Gamma^s)^{\mathcal{H}_f} \oplus (\Gamma^u)^{\mathcal{H}_f}.$$  

By Lemma 2.2, there is a continuous surjection $\tilde{h} : X \to \tilde{A}$ satisfying $\pi \tilde{h} = h$. For given $\tilde{p} = (\tilde{p}_0) \in \tilde{A}$, there exists $x \in X$ such that $\tilde{h}(x) = \tilde{p}$ and $h(x) = p_0$. Define subspaces of $T_{p_0}M$ by

$$E^{s}_{\tilde{h}}(p_0) := \{\xi(x) \mid \xi \in (\Gamma^s)^{\mathcal{H}_f}\},$$

$$E^{u}_{\tilde{h}}(p_0) := \{\xi(x) \mid \xi \in (\Gamma^u)^{\mathcal{H}_f}\}.$$
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\[ v = \xi \cdot \zeta^n, \quad \xi \in \mathcal{H}_f, \quad \zeta^n \in \mathcal{H}_g, \quad \mathcal{H}_f \cap \mathcal{H}_g = \{0\}. \quad (2.8) \]

For the purpose of showing

\[ E^s_u(\mathcal{P}(\mathcal{P})) \cap E^u_s(\mathcal{P}(\mathcal{P})) = \{0\}, \]

let \( v \in E^s_u(\mathcal{P}(\mathcal{P})) \cap E^u_s(\mathcal{P}(\mathcal{P})) \). Since \( v \in E^s_u(\mathcal{P}(\mathcal{P})) \), it follows that there is \( \xi \in \mathcal{H}_f \) satisfying \( \xi(x) \equiv v \). Here, we can take \( \xi \) satisfying \( ||v|| = ||\xi|| \). Then,

\[ ||T^nfuv|| = ||T^nfuv|| = ||T^nfuv(\sigma^n(x))|| = ||(T^n\mathcal{H}_f)^n\xi(\sigma^n(x))|| \leq ||(T^n\mathcal{H}_f)^n\xi|| \leq C\gamma^n||\xi|| = C\gamma^n||v||, \quad n > 0. \quad (2.9) \]

Since \( v \in E^s_u(\mathcal{P}(\mathcal{P})) \), it follows that there is \( \xi \in \mathcal{H}_f \) satisfying \( \xi(x) = v \). For a function \( a : X \to \mathbb{R} \) we have

\[ a(x) \cdot ((T^n\mathcal{H}_f)^n\xi) (x) = a(x) \cdot T^nfuv = T^nfuv(\lambda(x) \cdot \xi(\sigma^n(x))). \]

It follows that we can take \( \xi \) satisfying the further condition

\[ ||(T^n\mathcal{H}_f)^n\xi|| = ||(T^n\mathcal{H}_f)^n\xi(\sigma^n(x))|| \]

for a fixed \( n \). Then, for any \( n \geq 1 \) we have

\[ ||v|| \leq ||\xi|| \leq C\gamma^n||T^n\mathcal{H}_f^n\xi|| \leq C\gamma^n||T^n\mathcal{H}_f(\sigma^n(x))|| = C\gamma^n||T^n\mathcal{H}_f^n||, \quad n > 0 \quad (2.10) \]

Since \( v \neq 0 \) cannot satisfy both (2.8) and (2.9), it follows \( T_0M = E^s_u(\mathcal{P}(\mathcal{P})) \oplus E^u_s(\mathcal{P}(\mathcal{P})) \).

Therefore, by (2.9) and (2.10), \( A \) is a hyperbolic set with skewness \( \eta \).

The following proposition is an endomorphism version of [K, p. 57, Theorem], and is included in the theorem in [P1], which F. Przytycki mentioned without proof.

2.6 Proposition. ([P1, Theorem 1, 13]). Let \( f : U \to M \) be a \( C^r \) map from an open subset \( U \) of \( M \), \( \sigma : X \to X \) be a homeomorphism, and \( h : X \to U \) be a continuous map satisfying \( h\sigma = fh \). Suppose \( A = h(X) \) is a hyperbolic set of \( f \) with skewness \( \eta_0 < 1 \). Then, for any \( \eta \) with \( \eta_0 < \eta < 1 \) and any \( \epsilon > 0 \) there is a neighbourhood \( N_0 \) of \( f \) in \( C^r(U, M) \), \( r \geq 1 \), satisfying the following condition: For any \( g \in N_0 \) there exists uniquely a continuous map \( h_\eta : X \to U \) such that \( g\eta_\eta = h_\eta \sigma \) and that \( h_\eta \) is a hyperbolic fixed point of \( \mathcal{H}_f \) with skewness \( \eta \) and \( (h_\eta, h) < \epsilon \).

Proof: By Proposition 2.5 (ii), \( h \) is a hyperbolic fixed point of \( \mathcal{H}_f \) with skewness \( \eta_0 \). By Proposition 2.5 (i), \( \mathcal{H}_f \in C^r(C^r(X, U), C^r(X, M)) \) depends continuously on \( g \in C^r(U, M) \). By Lemma 2.1, we may assume that the hyperbolic fixed point \( h \) of \( \mathcal{H}_f \) satisfies (2.6) and (2.7) with \( C = 1 \). Then, by [HP, § 4], we have a unique hyperbolic fixed point \( h_\eta \) of \( \mathcal{H}_f \) with skewness \( \eta \) and \( (h_\eta, h) < \epsilon \).

Let \( C^b(L, M) \) be the space of all bounded, possibly discontinuous maps from \( L \) to \( M \) with the topology induced from \( \sup_{x \in L} d(f(x), g(x)) \), \( f, g \in C^b(L, M) \). For \( h \in C^b(L, M) \), the space \( C^b(L, M) \) has the manifold chart \( \varphi : N \to \Gamma^b(h^*(\tau)), \) the space of all bounded, possibly discontinuous sections of \( h^*(\tau) \). \( \varphi \) is defined similarly as \( \Gamma^b(h^*(\tau)) \) case.
Let \( f: U \to M \) be a \( C^r \) map from an open subset \( U \) of \( M \). Let \( \sigma: X \to X \) be a homeomorphism on a compact topological space \( X \). Define \( \mathcal{H}_f: C^0(X, U) \to C^0(X, M) \) similarly as \( \mathcal{H}_f \). The proof of the following two propositions is more easy than that of Propositions 2.5 and 2.6.

**2.5' Proposition.** (i) \( \mathcal{H}_f \) is a \( C^r \) map, if \( f \) is \( C^r \). Furthermore, the map \( C^r(U, M) \to C^r(C^0(X, U), C^0(X, M)) \), defined by \( f \mapsto \mathcal{H}_f \), is continuous.

(ii) If \( h \in C^0(X, U) \) is a fixed point of \( \mathcal{H}_f \), then \( \Lambda = h(X) \) is an invariant set of \( f \). Furthermore, \( \Lambda \) is a hyperbolic set of \( f \) with skewness \( \eta \), if and only if \( h \) is hyperbolic fixed point of \( \mathcal{H}_f \) with skewness \( \eta \).

**2.6' Proposition.** Let \( f \) and \( \sigma \) be as above, and \( h := X \to U \) be a bounded, possibly discontinuous map satisfying \( h \circ f = \sigma \circ h \). Suppose \( \Lambda := h(X) \) is a hyperbolic set of \( f \) with skewness \( \eta_0 < 1 \). Then, for any \( \eta \) with \( \eta_0 < \eta < 1 \) and any \( \epsilon > 0 \) there is a neighbourhood \( N_0 \) of \( f \) in \( C^r(U, M) \), \( r \geq 1 \), satisfying the following condition: For any \( g \in N_0 \), there exists uniquely \( h_g \in C^0(X, U) \) such that \( h_g \circ h = h \circ h_g \) and that \( h_g \) is a hyperbolic fixed point of \( \mathcal{H}_f \) with skewness \( \eta - \epsilon \) and \( (h_g, h) < \epsilon \).

Let \( A \subseteq U \) be an invariant set of \( f: U \to M \). The local stable set for \( q \in A \) is defined by
\[
W^s_f(q) := \{ q \in U \mid (f^n(q), f^n(q)) \leq \epsilon \}, \quad \epsilon > 0,
\]
and the local unstable set with size \( \epsilon \) for a backward \( f \)-orbit \( (p_i)_{i \geq 0} \) of \( A \) is defined by
\[
W^u_f((p_i)_{i \geq 0}) := \{ q \in U \mid d(p_i, q_i) \leq \epsilon \}, \quad \epsilon > 0.
\]
Since a backward orbit \( (p_i)_{i \geq 0} \) and an orbit \( \tilde{p} := (p_i)_{i \in \mathbb{Z}} \) have correspondence each other, we put \( W^s_f(\tilde{p}) := W^s_f((p_i)_{i \geq 0}). \)

In our case, let \( f: U \to M \) be a \( C^r \) map, \( \sigma: X \to X \) a homeomorphism on a compact space, and \( h: X \to U \) a continuous map such that \( f \circ h = h \circ \sigma \). Suppose that \( \Lambda := h(X) \) is a hyperbolic set of \( f \).

If \( g: U \to M \) is sufficiently \( C^1 \) near \( f \), then by Proposition 2.6, \( \mathcal{H}_g \) has a unique hyperbolic fixed point of \( \mathcal{H}_g \). By Proposition 2.5, \( A_g := h_g(X) \) is a hyperbolic set of \( g \). It follows that for any \( g \)-orbit \( \tilde{p} = (p_i), \ p_0 = p, \) in \( A_g \) we have the splitting as (2.1):
\[
T_p M = E^u_g \tilde{p}(p) \oplus E^s_g \tilde{p}(p).
\]
We call \( W^s_g(p) \) and \( W^u_g(\tilde{p}) \) a local stable manifold and local unstable manifold, respectively, since these sets are manifolds by the following proposition:

**2.7 Proposition (Local stable manifold and local unstable manifold theorem).** Suppose that \( f: U \to M, \ \sigma: X \to X \) and \( h: X \to U \) satisfy the same condition as in Proposition 2.6. Let \( N_0 \) be the neighbourhood of \( f \) obtained by Proposition 2.6. Then, for any \( \epsilon > 0 \) and \( \eta \) with \( \eta_0 < \eta < 1 \), there exist numbers \( a > 0, b > 0 \) and a neighbourhood \( N_1 \) of \( f \) in \( N_0 \) satisfying the following condition for any \( g \in N_1 \):

(i) For any \( g \)-orbit \( \tilde{p} = (p_i), \ p_0 = p, \) in \( A_g := h_g(X) \), there are a local stable manifold \( W^s_{g \sigma}(p) \) and a local unstable manifold \( W^u_{g \sigma}(\tilde{p}) \) of class \( C^r \) tangent to \( E^u_{g \sigma}(p) \) and \( E^s_{g \sigma}(p) \) at \( p \), respectively.

(ii) \( \exp_{g \sigma}(W^s_{g \sigma}(p)) \) is the graph of a map \( G^s_{g \sigma}: B^s_{g \sigma} \to E^u_{g \sigma}(p) \), where \( B^s_{g \sigma} \) is a neigh-
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(3) The $C^r$ manifolds $W_{s,a}^\pm(p)$ and $W_{u,a}^\pm(p)$ defined continuously on $\hat{p} \in A_{\theta}$.

(iv) The Lipschitz constants, Lip $G_{s,a}^\pm$ and Lip $G_{u,a}^\pm$ are smaller than $\epsilon$. The angle of $W_{s,a}^\pm(p)$ and $W_{u,a}^\pm(p)$ at $p$ are greater than $\theta$.

(v) Let $W_{s,a}^\pm := \{ q \in W_{s,a}^\pm(p), \ p \in A_{\theta} \}$ and $W_{u,a}^\pm := \{ q \in W_{u,a}^\pm(\hat{p}), \ \hat{p} \in \tilde{A}_0 \}$. Then, $g$ contracts $W_{s,a}^\pm$ in itself with Lipschitz constant $< \eta$, and $g$ maps a subset in $\text{Int} W_{s,a}^\pm$ expansively onto $W_{u,a}^\pm$ with Lipschitz constant $> \eta^{-1}$.

**Proof:** Let $(N_h, \varphi)$ be a manifold chart with center $h$, where $h \in N_h \subset C^0(X, U)$ and $\varphi : N_h \to \Gamma^0(h^{*}T)$. Let $B_c$ be the correction of vectors $v \in TM$ such that $||v|| < c$, for a number $c > 0$. By Proposition 2.5(i), there is a neighbourhood $N \subset N_h$ of $f$ in $C^r(U, M)$ such that for any $g \in N$ the map 

$$H_g := \varphi(\varphi^{-1} : \Gamma^0(h^{*}B_c) \to \varphi(\mathcal{H}_h)$$

is well-defined for sufficiently small $c > 0$ and that the map $C^r(U, M) \to C^r(\Gamma^0(h^{*}B_c), \varphi(N_h))$, defined by $g \to H_g$, is continuous. In fact, $g \to \mathcal{H}_g$ is continuous and $H_g$ is the representation of $\mathcal{H}_g$.

$\Gamma^0(h^{*}B_c)$ is an open subset of $\varphi(\mathcal{H}_h) \subset \Gamma^0(h^{*}T)$. Since $(DHf)_{\xi} = (Th_{\varphi_f})_{\xi} = (Tf)_{\varphi_f}$ for any $\xi \in \Gamma^0(h^{*}T)$, it follows that the 0-section in $\Gamma^0(h^{*}B_c)$ is the hyperbolic fixed point of $H_f$ with splitting (2.5). Since, by Proposition 2.6, there exists a unique hyperbolic fixed point $h_0$ of $\mathcal{H}_g$ with skewness $< \eta$ satisfying $\varphi h_0 = \varphi g$, it follows that there is a unique hyperbolic fixed point, $\xi_0 \in \Gamma^0(h^{*}B_c)$, of $H_0$ with skewness $< \eta$ satisfying $\varphi h_0 = \xi_0$. By the hyperbolicity, $\mathcal{H}_0$ is locally diffeomorphic at $h_0$ and $H_0$ is an isomorphism. Recalling $h^{*}T = E^u \oplus E^s$, put $B^u := E^s \cap B_a$ and $B^s := E^u \cap B_a$ for any $a > 0$.

By the unstable manifold theorem and the stable manifold theorem for a poing [HP, Theorems 2.3 and 2.4], there are a neighbourhood $N_1 \subset N$ of $f$ and a number $a$ with $0 < a < c$ satisfying the following condition:

1. There is a unique map $G^u_{s,a} : \Gamma^0(h^{*}B^u) \to \Gamma^0(h^{*}B^u)$ whose graph in $W_{s,a}^u := \bigcap_{n \geq 0} (H_0)^n(\Gamma^0(h^{*}B^u))$. Moreover Lip $G^u_{s,a} < 1$ and $G^u_{s,a}$ is of class $C^r$. The assignment $g \to G^u_{s,a}$ is continuous as a map $N_1 \to C^r(\Gamma^0(h^{*}B^u), \Gamma^0(h^{*}B^u))$. The map $(H_0|W_{s,a}^u)^{-1} : W_{s,a}^u \to W_{s,a}^u$ is a contraction of $W_{s,a}^u$ with Lipschitz constant $< \eta$.

2. There is a unique map $G^u_{u,a} : \Gamma^0(h^{*}B^u) \to \Gamma^0(h^{*}B^u)$ whose graph is $W_{u,a}^u := \bigcap_{n \geq 0} (H_0)^n(\Gamma^0(h^{*}B^u))$. Moreover Lip $G^u_{u,a} < 1$ and $G^u_{u,a}$ is of class $C^r$. The assignment $g \to G^u_{u,a}$ is continuous as a map $N_1 \to C^r(\Gamma^0(h^{*}B^u), \Gamma^0(h^{*}B^u))$. The map $H_0|W_{u,a}^u : W_{u,a}^u \to W_{u,a}^u$ is a contraction with Lipschitz constant $< \eta$.

Recalling that $E^u$, $E^s$ and $E_a$ are subbundles of the bundle $h^{*}T \to X$, let $B^u_a$, $B^u_x$ and $B^u_x$ denote the fibres of $B^u_a$, $B^u_x$ and $B^u_x$ over the point $x \in X$, respectively. $G^u_{s,a}$ and $G^u_{u,a}$ induce the following $C^r$ maps:

$$G^u_{s,a} : B^u_{s,a} \to B^u_{s,a},$$

$$G^u_{u,a} : B^u_{u,a} \to B^u_{u,a},$$

since for any $v \in B^u_{s,a}$ (or $v \in B^u_{u,a}$) there is $\xi \in \Gamma^0(h^{*}B^u_a)$ ($\xi \in \Gamma^0(h^{*}B^u_x)$, respectively). The graphs $W_{s,a}$ and $W_{u,a}$ of $G^u_{s,a}$ and $G^u_{u,a}$ satisfy

$$- 77 - (27)$$
respectively.

Then
\[ W_{q,a}^\prime := \exp h(x)(W_{q,a}^\prime), \]
\[ W_{q,a}^u := \exp h(x)(W_{q,a}^u), \]
are \( C^r \) manifolds containing \( h_\theta(X) \).

Next, we show
\[ W_{q,a}^\prime = \{ q \mid \forall i \geq 0 \quad d(g^i h(x), g^i(q)) \leq a \} \quad (2.12) \]
\[ W_{q,a}^u = \{ q \mid \exists (q_i)_{i \geq 0}, \quad q_0 = q \quad \text{and} \quad \forall i \leq 0 \quad d(g^i h(x), q_i) \leq a \} \quad (2.13) \]

By the definition of \( W_{q,a}^\prime \), it is included in the right side of (2.12). By the definition of \( W_{q,a}^u \) and the injectivity of \( H_\theta \) we have
\[ H_\theta^{-1}(W_{q,a}^u) = \cap (H_\theta)^n(I^n(h^*B_a)) \subset W_{q,a}^u. \]

It implies that for any \( \xi \in W_{q,a}^u \), there is a sequence \( (\xi_i)_{i \geq 0} \) such that \( \xi_i \in W_{q,a}^\prime \), \( \xi_0 = \xi \), and \( H_\theta(\xi_i) = \xi_{i+1} \). Hence, we have that \( W_{q,a}^u \) is included in the right side of (2.13). To show the converse inclusions we consider the bounded sections, possibly discontinuous.

Similarly as (1) and (2) we have (1') and (2') as follows:

(1') There is a unique map \( G_\theta^\prime : \Gamma^\prime(h^*B_a) \rightarrow \Gamma^\prime(h^*B_a) \) whose graph is \( W_{q,a}^\prime = \cap (H_\theta)^n(I^n(h^*B_a)) \).

(2') There is a unique map \( G_\theta^u : \Gamma^u(h^*B_a) \rightarrow \Gamma^u(h^*B_a) \) whose graph is \( W_{q,a}^u = \cap (H_\theta)^n(I^n(h^*B_a)) \).

By (1) and (1') we have
\[ G_\theta = G_\theta^\prime \mid \Gamma(h^*B_a), \quad W_{q,a}^\prime = W_{q,a}^\prime \cap \Gamma(h^*B_a). \]

Similarly, we have
\[ G_\theta = G_\theta^u \mid \Gamma(h^*B_a), \quad W_{q,a}^u = W_{q,a}^u \cap \Gamma(h^*B_a). \]

Therefore,
\[ W_{q,a,x}^\prime = W_{q,a}^\prime \cap B_{q,a}, \quad W_{q,a,x}^u = W_{q,a}^u \cap B_{q,a}. \]

To show (2.12), let \( q \) a point satisfying
\[ \forall i \geq 0, \quad d(g^i h(x), g^i(q)) \leq a. \]

Define \( \xi_q \in \Gamma^h(h^*B_a) \) by
\[ \xi_q := \{ \exp_{h(q)}^{-1}(q) \}. \]

Since \( \xi_q \in \Gamma^h \), it follows \( \exp_{h(q)}^{-1} \in \Gamma^h \). This implies
\[ W_{q,a,x} \cap \{ q \} \quad \forall i \geq 0, \quad d(g^i h(x), g^i(b)) \leq a. \]

Therefore, we have (2.12). Similarly, (2.13) is satisfied.

For any \( h \)-orbit \( \tilde{p} = (p_t) \), \( h_\theta(X) \), there is a unique point \( x \in X \) satisfying \( h_\theta h^t(x) = p_t \), by Lemma 2.2(i). Therefore,
\[ W_{q,a}(\tilde{p}) = W_{q,a,x}, \quad W_{q,a}(p) = W_{q,a,x} \]

are local stable and unstable manifolds, respectively. Obviously, \( W_{q,a}(\tilde{p}) \) and \( W_{q,a}(p) \) are (28)
tangent to $E^{s}_{\gamma}(p)$ and $E^{s}_{\gamma}(\bar{p})$ of (2.11), respectively. (This implies that $E^{s}_{\gamma}(p)$ in (2.11) does not depend on $\bar{p}$ but $p$.) Therefore, (i), (ii) and (iii) are proved.

(iv) The assignments $x \mapsto W_{\gamma,\alpha}^{s}(x)$ and $x \mapsto W_{\gamma,\alpha}^{u}(x)$ are continuous as the mappings from $X$ to the space of $C^{r}$ embeddings of balls $B^{1} \subset \mathbb{R}^1$ and $B^{u} \subset \mathbb{R}^u$, respectively. By (i), $W_{\gamma,\alpha}^{s}(x)$ and $W_{\gamma,\alpha}^{u}(x)$ are tangent at $h_{\gamma}(x)$ to $E^{s}_{\gamma}(x) := E^{s}_{\gamma}(p)$ and $E^{u}_{\gamma}(x) := E^{u}_{\gamma}(\bar{p})$, respectively. Moreover, by [P1, Theorem 1.10] $E^{s}_{\gamma}(x)$ and $E^{u}_{\gamma}(x)$ depend continuously on $x \in X$. Therefore, for given $\epsilon > 0$ by taking a smaller, if necessary, we have $\text{Lip } G_{\gamma,\alpha}^{s} < \epsilon$ and $\text{Lip } G_{\gamma,\alpha}^{u} < \epsilon$. By the continuity of $E^{s}_{\gamma}(x)$ and $E^{u}_{\gamma}(x)$, there is $\theta > 0$ such that the angle of $W_{\gamma,\alpha}^{s}(x)$ and $W_{\gamma,\alpha}^{u}(\bar{p})$ at $p$ are greater than $\theta$ for any $\bar{p}$.

(v) is obvious by (1) and (2) above.

Applying $(X, \sigma)$ and $h$ to the inverse limit $(\bar{A}, \bar{f})$ of $(A, f)$ and $\pi_{0}$, respectively, we have the following:

2.8 Corollary. ([P1, Theorem 2.1, Proposition 2.2]) Suppose that $A$ is a hyperbolic set of a $C^{r}$ map $f: U \to M$ with skewness $\eta_{0} < 1$. Then, for any $\epsilon > 0$ and $\eta$ with $\eta_{0} < \eta < 1$, there exist numbers $\alpha > 0$, $\theta > 0$ and a neighbourhood $N_{1}$ of $f$ in $C^{r}(U, M)$ satisfying the same conditions (i) $\sim$ (v) in Proposition 2.7 for any $g \in N_{1}$.

2.9 Corollary. Suppose $A$ is a hyperbolic set of $f$. Let $x$ be any point in $A$ and $(x_{i})$ be any $f$-orbit satisfying $x_{0} = x$. Then, for the splitting, $T_{x} M = E_{s} \oplus E_{u}$, at $x$ associated with the orbit $(x_{i})$ (in Definition 1.1). $E_{i}$ depends on $z \in A$ and does not depend on $(x_{i})$ such that $x_{0} = x$.

§ 3 Axiom A Endomorphisms

Hereafter in this section we suppose that $f: M \to M$ is a $C^{r}$ endomorphism satisfying Axiom A, $r \geq 0$. Put $\Omega := \Omega(f)$. For $p \in \Omega$ the stable set is defined by

$$W^{s}_{\gamma}(p) = \{ q \in M | d(f^{i}(q), f^{i}(p)) \to 0 (i \to \infty) \}.$$

For a backward $f$-orbit $(p_{i})_{i \in \Omega}$ with $p_{0} \in \Omega(f)$ the unstable set is defined by

$$W^{u}_{\gamma}(p_{i})_{i \in \Omega} := \{ q \in M | \text{there exists an } f^{-i} \text{-orbit } (q_{i}) \text{ such that } q_{0} = q \text{ and } d(p_{i}, q_{i}) \to 0 (i \to -\infty) \}.$$

A stable or unstable set may not be an immersed manifold. Since, for any backward orbit $(p_{i})_{i \in \Omega}$, there exists only one extending orbit $\bar{p} = (p_{i})$ we may put $W^{u}_{\gamma}(p_{i})_{i \in \Omega} = W^{u}_{\gamma}(\bar{p})$. By Proposition 2.7, $W^{s}_{\gamma}(p)$ and $W^{u}_{\gamma}(\bar{p})$ are immersed $C^{r}$ manifold and continuously depend on $f \in C^{r}(M, M)$. These manifolds are sometimes denoted by $W^{s}(p)$ and $W^{u}(\bar{p})$.

The following two lemmas hold similarly as [Sm].

3.1 Lemma. If $\bar{p} = (p_{i})$ is a periodic orbit of $f$ and $U$ is an open set intersecting $W^{s}(p_{0})$, then $\bigcup_{m \geq 0} f^{m}(U) \supset W^{s}(\bar{p})$.

3.2 Lemma. (Cloud lemma). If $\bar{p} = (p_{i})$ is a periodic orbit with $p_{k} = p_{0}$ and, denoting $W^{u}_{p_{k}}(\bar{p}) := W^{u}(\bar{q}^{i} p_{i})$,
Let \( A \) be an invariant set of \( f \). The map \( f : A \to A \) is called an Anosov map with constant \( a > 0 \) if there exists \( a > 0 \) with the property that for any \( 0 < \epsilon \leq a \) there is a \( \delta > 0 \) such that, for any \( p \in A \) and any \( f \)-orbit \( q := (q_0) \) in \( A \), \( d(p, q_0, q_\delta) < \delta \) implies that \( W^s_f(p) \cap W^u_f(q) = \{ \text{exactly one point} \} \).

Let \( f \) be an Axiom A endomorphism. Let \( f : \Omega \to \Omega \) be the inverse limit of \( f|_\Omega : \Omega \to \Omega \), where we are denoting \( \Omega := \Omega(f) \). By Proposition 4.6, there is a neighborhood \( N_0 \) of \( f \) in \( C(M, M) \) such that for every \( q \in N_0 \) there exists a unique continuous map \( h_\Omega : \tilde{\Omega} \to M \) such that \( \Omega = h_\Omega(\tilde{\Omega}) \) is a hyperbolic set of \( \Omega \).

### 3.3 Proposition. (Existence of local product structure, [P1, proposition 3.9])

There exists a neighborhood \( N \) of \( f \) in \( C(M, M) \) such that, for every \( q \in N \), \( \Omega_q : \Omega_q \to \Omega_q \) is an Anosov map.

**Proof:** By Proposition 2.7, there are a neighborhood \( N_0 \) of \( f \) in \( C(M, M) \) and a constant \( a > 0 \) such that, for any \( q \in N_0 \) and any \( 0 < \epsilon \leq a \), there is a \( \delta > 0 \) satisfying the following condition, for any \( p \in \Omega_q \) and any \( f \)-orbit \( (q) := (q_0) \) in \( \Omega_q \), \( d(p, q_0) < \delta \) implies that \( W^s_f(p) \cap W^u_f(q) = \{ \text{exactly one point} \} \).

Let \( p \in \Omega_q \) and \( q = (q_0) \in \Omega_q \) such that \( d(p, q_0, q_\delta) < \delta \). Since \( \Omega = h_\Omega(\tilde{\Omega}) \) and \( \Omega = \Omega(f) \), it follows that there are \( \delta' \), \( q' \in \Omega \) such that \( h_\Omega(q') = p \) and \( h_\Omega(q') = q \) (by Lemma 2.2).

Since the correspondence \( q \mapsto h_\Omega(q) \) is continuous, it follows that the correspondences \( q \mapsto h_\Omega(q) \) and \( q \mapsto h_\Omega(q) \) are continuous. Since \( h_f = \pi_0 : \tilde{\Omega} \to \Omega \), and \( q_0 = \pi_0(q') \), it follows that there is a neighborhood \( N \) of \( f \) in \( N_0 \) such that \( f \in N_0 \) \( d(p, q_0) < \delta \) implies \( d(p', q') < \delta \), where \( p' := \pi_0(p') \) and \( q' := \pi_0(q') \) belong to \( \Omega \). Hence, \( W^s_f(p') \cap W^u_f(q') = \{ \text{exactly one point} \} \) is a point in \( \tilde{\Omega} \). Let \( \delta < \delta' \) be fixed.

Let \( p \in \Omega_q \) and \( q = (q_0) \in \Omega_q \) such that \( d(p, q_0, q_\delta) < \delta \). Since \( \Omega = h_\Omega(\tilde{\Omega}) \) and \( \Omega = \Omega(f) \), it follows that there are \( \delta' \), \( q' \in \Omega \) such that \( h_\Omega(q') = p \) and \( h_\Omega(q') = q \) (by Lemma 2.2).

Since the correspondence \( q \mapsto h_\Omega(q) \) is continuous, it follows that the correspondences \( q \mapsto h_\Omega(q) \) and \( q \mapsto h_\Omega(q) \) are continuous. Since \( h_f = \pi_0 : \tilde{\Omega} \to \Omega \), and \( q_0 = \pi_0(q') \), it follows that there is a neighborhood \( N \) of \( f \) in \( N_0 \) such that \( f \in N_0 \) \( d(p, q_0) < \delta \) implies \( d(p', q') < \delta \), where \( p' := \pi_0(p') \) and \( q' := \pi_0(q') \) belong to \( \Omega \). Hence, \( W^s_f(p') \cap W^u_f(q') = \{ \text{exactly one point} \} \) is a point in \( \tilde{\Omega} \). Let \( \delta < \delta' \) be fixed.

Let \( p \in \Omega_q \) and \( q = (q_0) \in \Omega_q \) such that \( d(p, q_0, q_\delta) < \delta \). Since \( \Omega = h_\Omega(\tilde{\Omega}) \) and \( \Omega = \Omega(f) \), it follows that there are \( \delta' \), \( q' \in \Omega \) such that \( h_\Omega(q') = p \) and \( h_\Omega(q') = q \) (by Lemma 2.2).

Since the correspondence \( q \mapsto h_\Omega(q) \) is continuous, it follows that the correspondences \( q \mapsto h_\Omega(q) \) and \( q \mapsto h_\Omega(q) \) are continuous. Since \( h_f = \pi_0 : \tilde{\Omega} \to \Omega \), and \( q_0 = \pi_0(q') \), it follows that there is a neighborhood \( N \) of \( f \) in \( N_0 \) such that \( f \in N_0 \) \( d(p, q_0) < \delta \) implies \( d(p', q') < \delta \), where \( p' := \pi_0(p') \) and \( q' := \pi_0(q') \) belong to \( \Omega \). Hence, \( W^s_f(p') \cap W^u_f(q') = \{ \text{exactly one point} \} \) is a point in \( \tilde{\Omega} \). Let \( \delta < \delta' \) be fixed.
Proof: We can follow the proof of [N, Chapter 6, Lemma 3] using Lemmas 3.1, 3.2 and Proposition 3.3.

3.5 Proposition. (Spectral decomposition theorem, [P1, P2]). There exists a unique decomposition of \( \Omega(f) \)
\[ \Omega(f) = \Omega_1 \cup \cdots \cup \Omega_k \]
into a finite number of disjoint closed invariant and transitive sets.

Proof: In the case of endomorphisms, we can also follow the proof of [Sm] or [N, Chapter 6, Theorem 1] using Lemma 3.4.

A sequence of points \( \{p_i\}_{i \in \mathbb{Z}} \) in \( M \) is called a \( \delta \)-pseudo-orbit of \( f \), if \( d(f(x_i), x_{i+1}) < \delta \) for every \( i \). A sequence of points \( \{p_i\}_{i \in \mathbb{Z}} \) is called to be \( \varepsilon \)-shadowed (or \( \varepsilon \)-traced) by a \( f \)-orbit \( \{q_i\} \) if \( d(p_i, q_i) < \varepsilon \) holds for any \( i \in \mathbb{Z} \). A map \( f \) is said to have the pseudo-orbit tracing property, if for every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that every \( \delta \)-pseudo orbit of \( f \) can be \( \varepsilon \)-shadowed by some \( f \)-orbit.

3.6 Lemma. (Shadowing Lemma). The endomorphism \( f \) has the pseudo-orbit tracing property on the non-wandering set \( \Omega(f) \).

Proof: Put \( \Omega := \Omega(f) \). By Proposition 2.7, \( f|\Omega : \Omega \to \Omega \) is an Anosov map. Then, by [Sa, Lemma 3] this map has the pseudo-orbit tracing property.

The following lemma is a direct conclusion of Corollary 2.8.

3.7 Lemma. (Expansiveness) There exist a \( C^1 \) neighbourhood \( N \) of \( f \) and a constant \( a > 0 \) such that for every \( f \in N \) if the \( g \)-orbits \( \{q_i\}_g \subset \Omega_g \) and \( \{q_i\}_f \subset M \) satisfy \( d(p_i, q_i) < a \) for any \( i \in \mathbb{Z} \), then \( p_i = q_i \) for any \( i \in \mathbb{Z} \).

The following proposition is the endomorphism version of [HP, Theorem 7.3].

3.8 Proposition. (Local maximality [P2, Lemma 1.3]) There exists a \( C^1 \) neighbourhood \( N \) of \( f \) and a compact neighbourhood \( U \) of \( \Omega = \Omega(f) \) with \( \Omega(f) \subset U \) such that for any \( g \in N \) the maximal invariant set \( \Lambda \) of \( g \) in \( U \) satisfies \( \Lambda = h_g(\bar{A}) \).

Proof: By Lemma 3.7, there exist a \( C^1 \) neighbourhood \( N \) of \( f \) and a constant \( a > 0 \) such that for any \( g \)-orbits \( \{q_i\}_g \) and \( \{q_i\}_f \) satisfying \( \tilde{q}_i \in h_g(\bar{A}) \), \( d(\tilde{q}_i, q_i) < a \) for all \( i \in \mathbb{Z} \) implies \( q_i = \tilde{q}_i \) for all \( i \in \mathbb{Z} \). By Lemma 3.6, there exists \( \delta \) with \( 0 < \delta < a \) such that any \( \delta \)-pseudo orbit of \( f \) in \( \Omega \) is \( a/3 \)-shadowed by an \( f \)-orbit \( \{p_i\} \) in \( \Omega \). Since \( f \) is uniformly continuous, it follows that there exists \( \varepsilon_1 < \delta / \varepsilon \) such that \( d(p, q) < \varepsilon_1 \) implies \( d(fp,fq) < \delta / 3 \).

Let \( U \) be a compact neighbourhood of \( \Omega \) satisfying \( \Omega(f) \subset \text{int} \ U \) and
\[ \forall q \in U, \ \exists p \in \Omega, \ d(p, q) < \varepsilon_1. \]
We may take the above neighbourhood \( N \) of \( f \) such that for any \( g \in N \) and any \( p \in M \),
\[ d(fp, q) < \varepsilon_2 := \min\{\frac{a}{3}, \frac{\delta}{3}\}. \]

Let \( g \in N \) be a given map and \( \{q_i\}_g \) be a given \( g \)-orbit in \( U \). By the definition of \( U \)
we have

\[ i \in \mathbb{Z}, \quad p_i' \in \Omega, \quad d(p_i', q_i) < \epsilon_1. \]

Hence, we have, for every \( i \in \mathbb{Z} \),

\[ d(f(p_i'), p_{i+1}) \leq d(f(p_i'), f(q_i)) + d(f(q_i), q_{i+1}) + d(q_{i+1}, p_{i+1}) < \frac{\epsilon_1}{3} + \frac{\epsilon_1}{3} + \frac{\epsilon_1}{3} = 3 \epsilon_1. \]

It follows that \( \{ p_i' \}_{i \in \mathbb{Z}} \) is a \( \delta \)-pseudo orbit of \( f \) in \( \Omega \). Therefore, \( \{ p_i' \}_{i \in \mathbb{Z}} \) in \( a/3 \)-shadowed by an \( f \)-orbit \( \{ p_i \} \) in \( \Omega \). Putting \( \bar{q}_i := h_\gamma(\{ p_i \}) \), we have

\[ d(\bar{q}_i, q_i) \leq d(p_i, p_i') + d(p_i', q_i) < \frac{\epsilon_1}{3} + \frac{\epsilon_1}{3} + \frac{\epsilon_1}{3} = 3 \epsilon_1. \]

Therefore, \( \bar{q}_i = q_i \) for all \( i \in \mathbb{Z} \). This implies \( \Lambda \subset h_\gamma(\bar{U}) \). Since the invariant set \( h_\gamma(\bar{U}) \) is included in \( U \) for every \( \gamma \) sufficiently \( C^1 \) near \( f \), it follows that \( h_\gamma(\bar{U}) \subset \Lambda \). \( \blacksquare \)

Let \( \Omega(f) = \Omega_1 \cup \cdots \cup \Omega_r \) be the spectral decomposition of \( f \). Let

\[ W^s(\Omega_i) := \bigcup_{p \in \Omega_i} W^s(p), \]

\[ W^u(\Omega) := \bigcup_{p \in \Omega} W^u(p). \] (3.2)

3.9 Lemma. For any \( l = 1, \ldots, r \) there is a neighbourhood \( U_l \) of \( \Omega_l \) satisfying

(i) \( \cap_{m \geq 0} f^{-m}(U_l) \subseteq W^s(\Omega_l) \), and

(ii) \( \cap_{m \geq 0} V_m \subseteq W^u(\Omega_l) \), where \( V_m \) is defined inductively by \( V_0 := U_l \) and \( V_{k+1} := U_l \cap f(V_k) \).

Proof: Similarly as the proof of Proposition 3.8, we can take \( U_l \) satisfying that for any \( \epsilon > 0 \) there is a \( \delta > 0 \) such that every \( \delta \)-pseudo orbit in \( U_l \) is \( \epsilon \)-shadowed by an \( f \)-orbit in \( \Omega_l \).

(i) Given \( q_0 \in \cap_{m \geq 0} f^{-m}(U_l) \) satisfies \( f^m(q_0) \in U_l \) for \( m \geq 0 \). Hence, there is a \( \delta \)-pseudo orbit \( \{ q_m \}_{m \in \mathbb{Z}} \) in \( U \) with \( q_m = f^m(q_0) \) for \( m \geq 0 \). Then, \( \{ q_m \} \) is \( \epsilon \)-shadowed by an \( f \)-orbit \( \{ p_m \}_{m \in \mathbb{Z}} \) in \( \Omega_l \). Put \( p := p_0 \). We have \( d(f^m(p), f^m(q_0)) = d(p_m, q_m) < \epsilon \) for every \( m \geq 0 \). Therefore \( q_0 \in W^s(p) \).

(ii) If \( q_0 \in \cap_{m \geq 0} V_m \), then there is a sequence \( \{ q_m \}_{m \leq 0} \) in \( U \) containing the given \( q_0 \) satisfying \( f(q_{m-1}) = q_m \). This sequence is extended to a \( \delta \)-pseudo orbit \( \{ q_m \}_{m \in \mathbb{Z}} \) in \( U \), which is \( \epsilon \)-shadowed by an \( f \)-orbit \( \tilde{p} = \{ p_m \}_{m \in \mathbb{Z}} \) in \( \Omega_l \). Since \( d(p_m, q_m) < \epsilon \) for every \( m \leq 0 \), it follows that \( q_0 \in W^u(\tilde{p}) \). \( \blacksquare \)

For \( \tilde{p} = \{ p_i \} \in \Omega_l \), let denote

\[ \tilde{W}_s(\tilde{p}) := \{ (q_i) \in \tilde{M} : d(p_i, q_i) \leq \epsilon \text{ for } i \leq 0 \}, \]

\[ \tilde{W}^u_{\text{open}}(\tilde{p}) := \{ (q_i) \in \tilde{M} : d(p_i, q_i) \leq \epsilon \text{ for } i \leq 0 \}. \]

For \( 0 < \delta < \epsilon \), let denote

\[ F^u(\epsilon, \delta) := \bigcup_{\tilde{p} \in \tilde{U}} (\tilde{W}_s(\tilde{p}) - \tilde{W}^u_{\text{open}}(\tilde{p})), \]

\[ F^u(\epsilon, \delta) := \pi_2(\tilde{F}^u(\epsilon, \delta)). \]

We have \( F^u(\epsilon, \delta) \subseteq \tilde{W}_s(\Omega_l) \). \( F^u(\epsilon, \delta) \) is compact by \([P2, \text{p. 65}]\); we can verify this also by using Proposition 2.7(iii). Let denote

\[ (32) \]
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\[ O_-(P_u(\varepsilon, \delta)) := \bigcup_{j \leq 0} \pi_0 f^j(\hat{P}_u(\varepsilon, \delta)), \]
\[ O_+(P_u(\varepsilon, \delta)) := \bigcup_{j \geq 0} \pi_0 f^j(\hat{P}_u(\varepsilon, \delta)) = \bigcup_{j \geq 0} f^j(P_u(\varepsilon, \delta)), \]
\[ O(P_u(\varepsilon, \delta)) := O_-(P_u(\varepsilon, \delta)) \cup O_+(P_u(\varepsilon, \delta)). \]

For \( \varepsilon > 0 \) and a sufficiently small \( \delta > 0 \), we have
\[ O_-(P_u(\varepsilon, \delta)) \supset W^s_\varepsilon(\Omega_1) \setminus \Omega_1, \quad (3.3) \]
\[ O(P_u(\varepsilon, \delta)) = W^u_\varepsilon(\Omega_1) \setminus \Omega_1. \quad (3.4) \]

In fact, for any small \( \varepsilon > 0 \) there is \( \delta > 0 \) satisfying \( \hat{f}(\hat{W}_1) \subset \hat{W}_1(\Omega_1) \) by Proposition 2.7(iii).

Then, we have
\[ f^\varepsilon(\Omega_1) \supset \hat{W}_1(\Omega_1) \supset f^\varepsilon(\Omega_1). \]

Hence, we have (3.3) as follows;
\[ O_-(P_u(\varepsilon, \delta)) = \pi_0 \big( \bigcup_{j \leq 0} f^j P_u(\varepsilon, \delta) \big) \]
\[ \supset \pi_0 \big( \hat{W}_1(\Omega_1) \setminus \bigcap_{j \leq 0} f^j \hat{W}_1(\Omega_1) \big) = W^s_\varepsilon(\Omega_1) \setminus \Omega_1. \]

(3.4) is a direct consequence of (3.3). We call \( P_u(\varepsilon, \delta) \) satisfying (3.3) a fundamental domain for \( \Omega_1 \).

3.10 Lemma. Let \( U_1 \) be the neighborhood of \( \Omega_1 \) obtained by Lemma 3.9. Suppose that there exists a compact set \( X \) in \( M \) satisfying

(i) \( f(X) \subset \text{int} X \),
(ii) \( \bigcup_{i \geq 0} f^i(X) \cup \Omega_1 \supset W^u_\varepsilon(\Omega_1) \).

Then, there exist \( m > 0 \) and a compact set \( Y \subset M \) such that

(i) \( f(Y) \subset \text{int} Y \),
(ii) \( \Omega_1 \subset \text{int} Y - f^{-m}(X) \subset U_1 \).

Proof: We can find a fundamental domain \( P_u \) for \( \Omega_1 \) with
\[ O_-(P_u) \subset U_1, \quad (3.5) \]
\[ \exists m > 0, \quad f^{-m}(X) \supset P_u, \quad (3.6) \]
by using the assumption (ii). Define two sets, \( P \) and \( Q \), by
\[ P := W^u(\Omega_1) \cup \bigcap_{j \geq 0} f^j(X), \quad Q := U_1 \cup f^{-m}(X). \]

Define the sequence of subsets \( Q_0, Q_1, \ldots \) by the induction; \( Q_0 := Q \), \( Q_j := f(Q_{j-1}) \cap U_1 \).

Sublemma:
\[ \bigcap_{j \geq 0} Q_j = P. \quad (3.7) \]

1° Proof of \( \bigcap_{j \geq 0} Q_j \supset P \): Since the assumption (i), \( f(X) \subset \text{Int} X \), implies \( f^{-m}(X) \supset \bigcup_{j \leq m} f^{-j}(X) \), we have \( f^{-m}X \supset O_+(P_u) \) by (3.6). Hence, by (3.5) we have
\[ Q_0 = U_1 \cup f^{-m}(X) \supset O(P_u). \]

Since it implies \( f(Q_0) \supset O(P_u) \), it follows that we have \( Q_1 = Q_0 \cup f(Q_0) \supset O(P_u) \). Inductively, we have
\[ \bigcap_{j \geq 0} f^j(Q) \supset O(P_u) = W^u(\Omega_1) - \Omega_1. \quad (3.8) \]

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Since $Q \supset f^{-m} X \supset X$, we have
\begin{equation}
\bigcap_{j \geq 0} f^j(Q) \supset \bigcap_{j \geq 0} f^j(X) \tag{3.9}
\end{equation}
By the definitions of $P$ and $Q$, (3.8) and (3.9) implies $\cap_{k \geq 0} Q_k \supset P$.

2° Proof of $\cap_{j \geq 0} Q_j \subset P$: Supposing that a point $q \in \cap_{j \geq 0} Q_j$ does not belong to $\cap_{j \geq m} f^j(X)$, we will show $q \in W^u(Q_j)$.

Since $q \in Q_j$ for any $j \geq 0$, it follows that for any $k \geq 0$ there is $p \in Q_k$ satisfying $f^k(p) = q$ and $f^j(p) \in Q_j$ $(j = 0, 1, \ldots, k)$.

Since $q \in \cap_{j \geq 0} f^j(X)$, there is $k$ such that the point $p$ obtained above for $k$ satisfies $p \in U_k - f^{-m}(X)$. In fact, we can take $k > m$ satisfying $q \in f^{k-m}(X)$.

For the point $p$ obtained in (a) we can assume $p \in \cap_{j \geq 0} Q_j$. In fact, since $q \in Q_{k+h}$ for any $h \geq 0$, it follows that there is $p_h \in Q_h$ satisfying $f^{k}(p_h) = q$ and $f^j(p_h) \in Q_j$ $(j = 0, 1, \ldots, k)$. We have $Q_0 \supset Q_1 \supset Q_2 \supset \ldots$. In fact, $Q_0 \supset Q_1 = Q_0 \cap f(Q_0)$; and $Q_j \supset Q_{j+1}$ implies $Q_{j+1} = Q_0 \cap f(Q_j) \supset Q_0 \cap f(Q_{j+1}) = Q_{j+2}$. Therefore, we have $p \in \cap_{j \geq 0} Q_j$.

Moreover, we have $p \in \cap_{j \geq 0} V_j$, where $V_j$ is the set defined in Lemma 3.9: Since $q = f^k(p) \in \cap_{j \geq 0} f^j(X)$, it follows that $p \in \cap_{j \geq 0} f^j(X)$. Similarly as above, for any $i \geq 0$ there is $p_i \in U_i - f^{-m}(X)$ such that $f^i(p_i) = p$ and $f^j(p_i) \in Q_j$ $(j = 0, 1, \ldots, i)$. It follows $f^i(p_i) \in U_i \cap f(U_i) = V_i$ since $f^i(p_i) \in \cap_{k \geq 0} f^k(X)$. And, inductively, we have $f^j(p_i) \in V_j$ for $j = 0, 1, \ldots, i$. Therefore, $p = f^i(p_i) \in V_i$ for any $i \geq 0$.

Hence, by Lemma 3.9 we have $p \in W^u(Q_j)$; and $q = f^k(p) \in W^u(Q_k)$. Therefore, we have $\cap_{j \geq 0} Q_j \subset P$.

We continue the proof of Lemma 3.10. By Sublemma above and P. Przytycki [P2, p. 64], there exists a compact $Y$ satisfying (i) and (ii) in Lemma 3.10.

3.11 Lemma. $M = W^s(Q_1) \cup \cdots \cup W^s(Q_r)$.

Proof: Let $W^s(Q_k) := \{ x \in M | f^m(x) \to \Omega_k \text{ as } m \to \infty \}$. Then, we have $M = \bigcup_{i \geq 1} W^s(Q_i)$ (see [N, p. 191, Corollary]). By Lemma 3.9(i) we get the desired decomposition.

We say that $f$ satisfies the no-cycle condition if there exists no sequence of numbers $j_1, \ldots, j_k (k \geq 1)$ such that
\[(W^s(Q_{j_1}) - Q_{j_k}) \cap (W^u(Q_{j_{k+1}}) - Q_{j_{k+1}}) \neq \emptyset\]
for $1 \leq h \leq k$ and $j_1 = j_k$. The following proposition is proved by F. Przytycki.

3.12 Proposition. ([Filtration [P2, Proposition 1.1]]) If $f$ satisfies the no-cycle condition, then for any family of compacci neighborhood $U_l$ of $\Omega_l$ there exist a finite sequence of compact set, $M_0, \ldots, M_r$, and a sequence of positive integers, $m_1, \ldots, m_r$, such that
\begin{enumerate}
  \item $\emptyset = M_0 \subset \cdots \subset M_r = M$,
  \item $f(M_l) \subset \text{int } (M_l)$ for every $l$,
  \item $\Omega_l \subset M_l - f^{-m_l}(M_{l-1}) \subset \text{int } U_l$.
\end{enumerate}
Proof: There is a simple order of the basic sets, $\Omega_0 \leq \cdots \leq \Omega_1 \leq \cdots \leq \Omega_r$ such that $\Omega_i \leq \Omega_j$ if $W^s(\Omega_i) \subset W^u(\Omega_j) = \emptyset$. The proof is by the induction on $l$.

1. $l = 0$: Since $M = \bigcup W^s(\Omega_i)$ and $W^u(\Omega_i) = \Omega_0$, it follows that $W^u(\Omega_i) \cap W^u(\Omega_i) = \emptyset$ for any $i > 0$. In Lemma 3.10, we let $X = \emptyset$ to get $M_0 = Y$, possessing all of the required properties by Lemma 3.10.

2. The induction step: Given $M_{i-1}$, we need only check
\[
\bigcup_{j \leq i} f^j(M_{i-1}) \cup \Omega_i \supset W^s(\Omega_i)
\]
(3.10)
to apply Lemma 3.10, with $X = M_{i-1}$. By Lemma 3.11 and the no-cycle condition we have
\[
W^u(\Omega_i) = \bigcup_{i=0}^r (W^u(\Omega_i) \cap W^u(\Omega_i))
= \bigcup_{i=1}^{i+1} (W^u(\Omega_i) \cap W^u(\Omega_i))
\]
(3.11)
Since $W^u(\Omega_i) \cap W^u(\Omega_i) = \Omega_i$ and
\[
\bigcup_{j \leq i} f^j(M_{i-1}) \supset W^s(\Omega_i) \quad (\forall i \leq l),
\]
We have (3.10) from (3.11). We apply Lemma 3.10 to get $M_i$ satisfying (1) ~ (3).

Let $X$ be compact metric space and $h$ be a continuous map from $X$ to itself. An $\epsilon$-
chain from $x$ to $y$ in $X$ is a finite sequence of points $\{x_0, x_1, \ldots, x_n\}$ of $X$ with $x = x_0$, $y = x_n$, and $d(h^i x_{i-1}, x_i) < \epsilon$ for $i = 1, \ldots, n$. We let $R(x)$ denote the set of $y \in X$ such that $y$ is a chain recurrent from $x$ to $y$. We say $x$ can be chained to $y$ if $y \in R(x)$ for every $\epsilon > 0$, and $x$ is chain recurrent if $x$ can be chained to $x$. The set of all chain recurrent points is denoted by $R(h)$. We have $\Omega(h) \subset R(h)$. Denoting $\Omega := \Omega(h)$, we say $h$ does not permit $\Omega$-explosions, if for every $\epsilon > 0$ there is a neighborhood $N$ of $h$ in $C^r(X, X)$ such that if $g \in N$ then for any $x \in \Omega(g)$, $d(x, \Omega) < \epsilon$. The following lemma is proved by L. Blok and E. Franke.

3.13 Lemma [BF]. If $\Omega(h) = R(h)$ then $h$ does not permit $\Omega$-explosions.

3.14 Proposition. If $f$ is an Axiom A endomorphism satisfies the no-cycle condition, then $f$ does not permit $\Omega$-explosions.

Proof: By Proposition 3.12 $f$ has the filtration adapted to the spectral decomposition. It follows that $R(f) = \Omega(f)$, so that $R(f) = \Omega(f)$. Therefore, by Lemma 3.13 $f$ does not permit $\Omega$-explosions.

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