Normality of blowing-up

By

Shiro GOTO and Kikumichi YAMAGISHI*

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1. Introduction.

Let $A$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $d = \text{dim } A > 0$. Let $q = (a_1, a_2, \ldots, a_d)$ be a parameter ideal in $A$ and put $R = \bigoplus_{n \geq 0} q^n$, the Rees ring of $q$. In this note we shall explore when the scheme $\text{Proj } R$ is normal and our result is stated as follows:

**Theorem (1.1).** Suppose that depth $A > 0$. Then the following conditions are equivalent.

1. $\text{Proj } R$ is normal.
2. $A$ is a regular local ring and $\ell_A(q + \mathfrak{m}^2/\mathfrak{m}^3) \geq d - 1$.

When this is the case, the ring $R$ is normal. (Here $\ell_A(q + \mathfrak{m}^2/\mathfrak{m}^3)$ stands for the length of the $A$-module $q + \mathfrak{m}^2/\mathfrak{m}^3$.)

In [6] the second author tackled with this theme and mentioned the equivalence of (1) and (2) in (1.1) with a rather strong assumption that $A$ is Cohen-Macaulay (cf. [6, Ch. 4, (1.3)]); our theorem guarantees his assumption can be replaced by the weaker one that depth $A > 0$.

We will prove Theorem (1.1) in the next section. As is noted in (1.1), the ring $R$ is normal if (and only if) $\text{Proj } R$ is normal and depth $A > 0$. The normality of $R$ itself is characterized in divers manners; especially, appealing to a recent result of J. Watanabe [7] on $\mathfrak{m}$-full ideals, we can prove that $R$ is normal if and only if $q$ is $\mathfrak{m}$-full. As the fact may have its own significance, in Section 3 we will discuss this subject a little more closely.

Throughout this note let $A$ denote a Noetherian local ring with maximal ideal $\mathfrak{m}$. We assume that $\text{dim } A = d > 0$ and fix a parameter ideal $q = (a_1, a_2, \ldots, a_d)$ in $A$. Let $R = \bigoplus_{n \geq 0} q^n$ be the Rees ring of $q$.

2. Proof of Theorem (1.1).

Let $B = A[x/a_i | x \in q]$ and $P = \mathfrak{m}B$. To begin with we note

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**Proposition (2.1).** (1) \( \dim B = d. \)
(2) \( P \) is a height 1 prime ideal of \( B \) and \( P = \sqrt{a_1}B. \)
(3) The elements \( a_i/a_1 \mod P(1 \leq i \leq d) \) of \( B/P \) are algebraically independent over \( A/m. \)

Let \( \mathfrak{p} \in \text{Spec} \ A \) with \( \mathfrak{p} \not
\supseteq a_1. \) We put
\[ I(\mathfrak{p}) = \mathfrak{p} A[1/a_1] \cap B. \]
Then \( I(\mathfrak{p}) \in \text{Spec} \ B, \ I(\mathfrak{p}) \cap A = \mathfrak{p}, \) and \( B/I(\mathfrak{p}) = A_{\mathfrak{p}}. \) Let \( x^* = x \mod \mathfrak{p} \) for each \( x \in A. \)

**Lemma (2.2).** \( B/I(\mathfrak{p}) = A[\mathfrak{p}[x^*/a_1^*]|x \in \mathfrak{a}] \) as \( A \)-algebras. In particular \( \dim B/I(\mathfrak{p}) \leq \dim A/\mathfrak{p}. \)

**Proof.** By definition we get an embedding \( B/I(\mathfrak{p}) \subseteq A/\mathfrak{p}[1/a_1^*] \) of \( A \)-algebras, whose image coincides with \( A[\mathfrak{p}[x^*/a_1^*]|x \in \mathfrak{a}] \). As \( \dim A[\mathfrak{p}[x^*/a_1^*]|x \in \mathfrak{a}] \leq \dim A/\mathfrak{p} \) by the dimension formula, the second assertion follows from the first.

**Corollary (2.3).** (1) \( B = \{ I(\mathfrak{p}) | \mathfrak{p} \in \text{Ass} \ A \text{ and } \mathfrak{p} \not
\supseteq a_1 \}. \)
(2) \( \{ I \in \text{Spec} \ B | \dim B/I = d \} = \{ I(\mathfrak{p}) | \mathfrak{p} \in \text{Spec} \ A \text{ and } \dim A/\mathfrak{p} = d \} = \{ I \in \text{Spec} \ B | I \not
\subseteq I \}. \)

**Proof.** Let \( I \in \text{Spec} \ B \) and put \( \mathfrak{p} = I \cap A. \) Then if \( \mathfrak{p} \not
\supseteq a_1, \) we have \( I = I(\mathfrak{p}) \) as \( A[1/a_1] = B[1/a_1]. \) Hence we get Assertion (1), because \( a_1 \) is \( B \)-regular and \( A_{\mathfrak{p}} = B_{I(\mathfrak{p})}. \)

Consider Assertion (2). First of all take \( I \in \text{Spec} \ B \) with \( \dim B/I = d. \) Then as \( B/I = 0, \) we may write \( I = I(\mathfrak{p}) \) with \( \mathfrak{p} = I \cap A. \) Notice \( \dim B/I = d \leq \dim A/\mathfrak{p} \) by (2.2) and we get \( \dim A/\mathfrak{p} = d. \) Conversely let \( \mathfrak{p} \in \text{Spec} \ A \) and assume \( \dim A/\mathfrak{p} = d. \) Then \( \mathfrak{p} \not
\supseteq a_1 \) clearly. We put \( I = I(\mathfrak{p}) \). Recall that \( B/I = A[\mathfrak{p}[x^*/a_1^*]|x \in \mathfrak{a}] \) as \( A \)-algebras and we see by (2.1) that the ring \( B/P + 1 = (B/I)/\mathfrak{m}(B/I) \) is a polynomial ring with \( d-1 \) variables over the field \( k = A/m. \) Hence the canonical epimorphism \( B/P \longrightarrow B/P + 1 \) of \( k \)-algebras must be an isomorphism, because \( B/P \) and \( B/P + 1 \) are \( k \)-isomorphic; thus \( P \supseteq I. \) Finally let \( I \in \text{Spec} \ B \) with \( I \not
\subseteq P. \) Then \( \dim B/I = d, \) as \( \dim B/P = d-1 \text{—— this completes the proof of Assertion (2).} \)

Let \( e(A) \) (resp. \( e(B_P) \)) denote the multiplicity of \( A \) (resp. \( B_P \)).

**Lemma (2.4).** \( e(B_P) \geq e(A). \)

**Proof.** Let \( h : A[T_2, T_3, ..., T_d] \longrightarrow B \) be the \( A \)-algebra map defined by \( h(T_i) = a_i/a_1 \) \( (2 \leq i \leq d), \) where \( T_2, T_3, ..., T_d \) are indeterminates over \( A. \) Let \( K = \text{Ker} \ h \) and \( f_i = a_i T_i - a_i \) \( (2 \leq i \leq d), \) then \( K \supseteq (f_2, f_3, ..., f_d). \) Notice \( a_i^{n+1} K \subseteq (f_2, f_3, ..., f_d) \) for some integer \( n \geq 1, \) because \( A[1/a_1] = B[1/a_1]. \) Now let \( C = A[T_2, T_3, ..., T_d] \) \( M = \text{m}(A[T_2, T_3, ..., T_d]) \) and consider the exact sequence
\[ 0 \longrightarrow L \longrightarrow C/(f_2, f_3, ..., f_d)C \longrightarrow B_P \longrightarrow 0 \]
of \( C \)-modules. Then as \( a_i^{n+1} L = (0) \) and as \( f_2, f_3, ..., f_d \) form a system of parameters for the local ring \( C, \) we have \( \ell_C(L) < \infty \) and therefore
\[ e(B_P) = e(C/(f_2, f_3, ..., f_d)C). \]
Recalling that \( e(C/(f_2, f_3, ..., f_d)C) \geq e(C) \), we get the required inequality \( e(B_P) \geq e(A) \), as \( e(C) = e(A) \).

We say that \( A \) is unmixed if \( \dim A^\vee / P = d \) for any \( p \in \Ass A \), where \( \hat{A} \) denotes the completion of \( A \). We shall use the following criterion of regularity.

**Proposition (2.5).** ([4, (40.6)]) A is a regular local ring if and only if \( e(A) = 1 \) and \( A \) is unmixed.

The next result (2.6) is a key theorem in this paper.

**Theorem (2.6).** Suppose that \( A \) is unmixed. Then the following conditions are equivalent.

1. \( A \) is a regular local ring and \( \ell_A(q + m^2/m^2) \geq d - 1 \).
2. \( B_P \) is a DVR.
3. \( \Proj R \) is normal.

**Proof.** \((3) \implies (2)\) Since \( \Spec B \) appears as one of the affine charts of \( \Proj R \), this implication is clear.

\((2) \implies (1)\) As \( e(A) = 1 \) by (2.4), we get \( A \) is regular (cf. (2.5)). We will prove that \( \ell_A(q + m^2/m^2) \geq d - 1 \). Let us maintain the same notation as in Proof of (2.4).

Notice that \( K = (f_2, f_3, ..., f_d) \) in our case, since \( a_1, a_2, ..., a_d \) is an \( A \)-regular sequence. Hence \( f_2, f_3, ..., f_d \) is a part of a minimal system of generators for the maximal ideal \( mC \) of \( C \), because \( B = C/(f_2, f_3, ..., f_d)C \) is a DVR by our assumption. Thus \( \ell_A(q + m^2/m^2) = \ell_C(qC + m^2C/m^2C) \geq d - 1 \), as \( qC = (a_1, f_2, ..., f_d)C \).

\((1) \implies (3)\) As the ring \( R \) is Cohen-Macaulay (cf. [1]), the scheme \( \Proj R \) satisfies the condition \( (S_2) \). So it is enough to check that all the rings \( A[x/a_i | x \in q] \) \((1 \leq i \leq d)\) satisfy the condition \( (R_i) \). We may assume without loss of generality that \( i = 1 \). Let \( I \in \Spec B \) with \( \dim B_I = 1 \). Then if \( I \ni a_1 \), \( B_I \cong A_p \) where \( p = I \cap A \) and \( B_I \) is a DVR in this case. Suppose that \( I \ni a_1 \). Then we get \( I = P \) by (2.1) (2). We must show that \( B_P \) is a DVR.

First of all notice that \( m = (a_1, ..., \hat{a}_i, ..., a_d, b) \) for some \( 1 \leq i \leq d \) and \( b \in m \), because \( \ell_A(q + m^2/m^2) \geq d - 1 \). We put \( J = bB_P \). Assume \( i \geq 2 \) and write \( a_i = \sum a_jx_j + by \) with \( x_j, y \in A \). Then as \( a_i = a_i - b \), we get

\[ a_i(a_i/a_i - \sum a_jx_j - x_i) \in J. \]

Hence \( a_i \in J \), because

\[ a_i/a_i - \sum a_jx_j - x_i \in P \]

(c.f. (2.1) (3)). We can similarly prove that \( a_i \in J \) for the case \( i = 1 \) too. Thus \( mB_P = bB_P \), which guarantees that \( B_P \) is a DVR. This completes the proof of (2.6).

**Remark (2.7).** Unless \( A \) is unmixed, the implication \([2) \implies (1)\] in (2.6) is not true in general even though \( A \) is an integral domain and \( B \) is a regular ring. In
fact according to M. Nagata [4], there exist a Noetherian local integral domain $A$ of dim $A=2$ and a system $a, b$ of parameters for $A$ which satisfy the following conditions:
(1) $A$ is not a regular ring;
(2) $B=A[b/a]$ is a regular ring.

Proof. Take a Noetherian local integral domain $A$ of dim $A=2$ so that
(1) the normalization $\bar{A}$ of $A$ is a regular ring and only has two maximal ideals, say $M$ and $N$;
(2) $m=M\cap N$;
(3) $A$ contains elements $x$ and $z$ such that $M=(x-1, z)A$, $N=x\bar{A}$, $z\in N$, and $\bar{A}=A+Ax$.

(Such rings $A$ must exist, see [4, p. 204].) Then $A$ is not regular as $A\approx \bar{A}$. We put $a=xz$ and $b=x(x-1)$. Then $a, b$ form a system of parameters in $A$. Let us check that $B=A[b/a]$ is a regular ring.

Recall that $z\in m=M\cap N$. Then we see $B\supset \bar{A}$, as $B$ contains $b/a=(x-1)/z$ and
as $\bar{A}=A+Ax$ by (3); hence $B=\bar{A}[<(x-1)/z]$. Let $Q$ be a prime ideal of $B$ and put $p=Q\cap \bar{A}$. If $Q\ni z$, then $Q$ contains $x-1=z(x-1)/z$ and so we have $p=M$ by (3). Therefore we get $B_p=\bar{A}_M[<(x-1)/z]$, which is a regular ring because $x-1, z$ is a regular system of parameters for $\bar{A}_M$ (cf., e.g., [2, (4.6)]). Hence the local ring $B_Q=(B_p)_Q$ is regular. If $Q\not\ni z$, then $B_Q=\bar{A}_p$ as $B[1/z]=\bar{A}[1/z]$ and we have $B_Q$ is a regular ring also in this case.

Corollary (2.8). The following conditions are equivalent.
(1) $B_p$ is a DVR.
(2) The completion $\hat{A}$ of $A$ contains a unique prime ideal $p$ such that dim $\hat{A}/p=d$.
Furthermore $\hat{A}/p$ is a regular local ring, $\ell_\hat{A}(\alpha\hat{A}+m^2\hat{A}+p/m^2\hat{A})\leq d-1$, and $\hat{A}_p$ is a field.

Proof. We put $C=\hat{A}[x/a_t|x\in q]$ and $Q=mc$. Then $C$ is a flat extension of $B$ as $C=\hat{A}\otimes B$. Notice $Q=PC$ and $P=Q\cap B$. Then we get $C_p$ is a DVR if and only if $B_p$ is; thus we may assume that $A$ is complete.

(1) $\implies$ (2) By (2.4) we get $e(A)=1$. Consequently by the formula
$$e(A) = \sum_{p\in \text{Spec} A} e(A_p) \cdot e(A/p)$$
(cf. [4, (23.5)]), we find that $A$ contains a unique prime ideal $p$ with dim $A/p=d$.
Furthermore $A/p$ is a regular local ring by (2.5), as $e(A/p)=1$. Clearly $A_p$ is a field. Now we will prove that $\ell_\hat{A}(\alpha+m^2+p/m^2+p)\geq d-1$.

Let $I=I(p)$. Then by (2.3) (2) we see $I\not\subset P$, whence $IB_P=(0)$ as $B_P$ is a DVR. On the other hand we have by (2.2) an isomorphism $B/I=\hat{A}[x^*/a_t|x\in q]$ of $A$-algebras. Thus the local ring $(\hat{A}/p[\alpha^*/a_t^*]|x\in q)_P$ is a DVR and we conclude by
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(2. 6) that \( \ell_A(q+m^2+p/m^2+p) \geq d-1 \).

(2) \( \Longrightarrow (1) \) Let \( I=I(p) \). Notice that \( I \) is, by (2. 3) (2), a unique prime ideal of \( B \) such that \( I \subseteq P \). Then we get that \( IB_P=(0) \), as \( B_P \) is a Cohen-Macaulay ring and \( B_P=A_P \) is a field. Recalling the isomorphism

\[
B/I=\frac{A[p[x^*/a^*|x \in q]},
\]

we have that \( B_P=IB_P \) is a DVR because by (2. 6) so is the local ring \( (A[p[x^*/a^*|x \in q]])_P \).

**Corollary (2. 9).** Assume that \( A \) is a homomorphic image of a Cohen-Macaulay ring and let \( a=a_1 \) be a non-zero divisor of \( A \). Then the following conditions are equivalent.

1. \( A \) is a normal ring.
2. (a) \( A[1/a] \) is a normal ring.
   (b) \( A \) contains a unique prime ideal \( p \) such that \( \text{dim } A/p = d \). Furthermore \( A/p \) is regular and \( \ell_A(q+m^2+p/m^2+p) \geq d-1 \).
   (c) For each \( Q \in \text{Ass } A \) with \( Q \neq p \), there is an integer \( N \geq 1 \) such that \( a^N \in q^{N+1}+Q \).

**Proof.** (1) \( \Longrightarrow (2) \) As \( A[1/a]=B[1/a] \), we see \( A[1/a] \) is a normal ring; hence \( A \) is reduced as \( A \subset A[1/a] \). Notice that \( B \) is integrally closed in the total quotient ring of \( A \), as it is normal. Then we get by (2. 2) and (2. 3) (1) an isomorphism

\[
B=\bigoplus_{Q \in \text{Ass } A} A/Q[x^*/a^*|x \in q]
\]

of \( A \)-algebras. Recall that \( e(A)=1 \) by (2. 4) and we find by the formula

\[
e(A)=\sum_{P \in \text{Spec } A} \ell(P) \cdot e(A/P)
\]

that \( A \) contains a unique prime ideal \( p \) of \( \text{dim } A/p = d \). Moreover \( A/p \) is, by (2. 5), a regular local ring because \( A/p \) is unmixed by our standard assumption. Let \( Q \in \text{Ass } A \) such that \( Q \neq p \). Then we get by the isomorphism (\#) that

\[
A/Q[x^*/a^*|x \in q] = \bigoplus_{Q \in \text{Ass } A} A/Q[x^*/a^*|x \in q],
\]

since \( P=mB \) is a prime ideal of \( B \) and since

\[
A/p[x^*/a^*|x \in q] = \bigoplus_{Q \in \text{Ass } A} A/p[x^*/a^*|x \in q]
\]

by (2. 1) (2). Hence we find that \( B_P=(A/p[x^*/a^*|x \in q])_P \) is a DVR and that the element \( a^*=a \mod Q \) is invertible in the ring \( A/Q[x^*/a^*|x \in q] \). Thus \( \ell_A(q+m^2+p/m^2+p) \geq d-1 \) by (2. 6) and \( a^N \in q^{N+1}+Q \) for some \( N \geq 1 \).

(2) \( \Longrightarrow (1) \) Let \( J \in \text{Spec } B \) and we will show that the local ring \( B_J \) is normal. If \( J \ni a \), this follows from Assumption (a) because \( B[1/a]=A[1/a] \). Assume \( J \ni a \), or equivalently \( J \supseteq P \).

**Claim.** \( J \ni 1(Q) \) for any \( Q \in \text{Ass } A \) such that \( Q \neq p \).

For, suppose that \( J \ni 1(Q) \) for some \( Q \in \text{Ass } A \) with \( Q \neq p \). Then as \( a^*=a \mod Q \)
is invertible in the ring $A/Q[x]/a^*|x\in q]$ (cf. Assumption (c)), we find by (2.2) that $I(Q)+P=B$ whence $J=B$—this is a contradiction.

By this claim and the embedding $B\subseteq \bigcap_{Q\in \text{Ass} A} B/I(Q)$ (recall that $\bigcap_{Q\in \text{Ass} A} I(Q) = (0)$ in $B$, see (2.3) (1)), we get that the ring $B_f$ appears as a local ring of $C=A/p[x]/a^*|x\in q]$. Hence $B_f$ is normal by (2.6).

**Example (2.10).** Let $S=k[[X, Y, Z, W]]$ be a formal power series ring over a field $k$ and let $I=(X)\cap (Y, Z)\cap (X-Y, Z-W)$ in $S$. We put $A=S/I$, $a=Z^2-X^2 \mod I$, $b=Y-X \mod I$, and $c=W-X \mod I$. Then $q=(a, b, c)$ is a parameter ideal in $A$ and $B=A/b^*a^*|x\in q]$. Hence $B_f$ is normal.

**Lemma (2.11).** Let $I=(b_1, b_2, \ldots, b_s)$ be an $m$-primary ideal of $A$. Then $A[x/b_i|x\in I]/m\cdot A[x/b_i|x\in I]$ for some $1\leq i\leq s$.

**Proof.** Assume the contrary and take an integer $N\geq 1$ so that $b_i^N\in mI^n$ for all $i$. Let $G=\bigoplus_{n=0}^N I^n/I^{n+1}$ and put $f_i=b_i \mod I^2$. Then as $f_i^N\in mG$, we find that all the $f_i$'s are nilpotent in $G$, whence $d=\dim G=0$—this is a contradiction.

In the situation of (2.11) we don't have always $A[x/b_i|x\in I]/m\cdot A[x/b_i|x\in I]$.

(For instance, consider $A=k[[t^2, t^3]]$ and $I=(t^2, t^3)$.) This is, of course, the case when $b_1, b_2, \ldots, b_s$ is a system of parameters in $A$, cf. (2.1).

We now prove Theorem (1.1).

**Proof of Theorem (1.1).** (2) $\implies$ (1) See (2.6).

(1) $\implies$ (2) According to (2.6) we have only to show that $A$ is unmixed. We put, as in Proof of (2.8), $C=A[x/a_1|x\in q]$ and $Q=mC$. Let $N$ be a maximal ideal of $C$ such that $N\supseteq Q$.

**Claim 1.** $\dim C_N/QC_N=d-1$.

**Proof.** The ideal $N/Q$ is maximal in the ring $C/Q$ and so we have that $\dim C_N/QC_N=d-1$, since $C/Q$ is a polynomial ring with $d-1$ variables over the field $A/m$, cf. (2.1) (3).

**Claim 2.** $\dim C_N/I=d$ for any $I\subseteq \text{Ass} C_N$.

**Proof.** Notice that $\text{Ass} a_1B/a_1B=(P)$ as $B$ is normal. Then we have $\text{Ass} cC/a_1C=(Q)$ as $B/a_1B\subseteq C/a_1C$. Let $I\subseteq \text{Ass} C_N$ and take $J\subseteq \text{Ass} cC_N/a_1C_N$ so that $J\supseteq I$ (this choice is possible as $a_1$ is $C_N$-regular, cf., e.g., [3, (15.D)]). Then we must have, as $\text{Ass} cC_N/a_1C_N=(Q_CN)$, that $J=Q_CN$ whence $\dim C_N/J=\dim C_N/QC_N=d-1$ by Claim 1.
Thus \( \dim C_N/I = d \) since \( J \subseteq I \).

Let us check that \( A \) is unmixed. Assume the contrary and pick \( p \in \text{Ass} \hat{A} \) so that \( \dim \hat{A}/p < d \). Then we get by (2.11)

\[
\hat{A}/p[x^*/a^* \mid x \in q] \cong \left( \hat{A}/p[x^*/a^* \mid x \in q] \right) \tag{\#}
\]

for some \( 1 \leq i \leq d \), where \( x^* = x \mod p \) for each \( x \in \hat{A} \). We may assume \( i = 1 \). Recall that the ideal \( q \) is generated by non-zerodivisors of \( A \), because depth \( A > 0 \) by our standard assumption. Hence we may further assume that \( a = a_1 \) is a non-zerodivisor of \( A \). Then as \( p \nmid a \), we get by (2.2) an isomorphism \( C/I = \hat{A}/p[x^*/a^* \mid x \in q] \) of \( \hat{A} \)-algebras, where \( I = I(p) \). According to (\#) this isomorphism guarantees that \( Q + I = C \), whence we may choose a maximal ideal \( N \) of \( C \) so that \( N \nsubseteq Q + I \). Then as \( I \in \text{Ass} C \) by (2.3) (1), we get \( \dim C_N/IC_N = d \) by Claim 2 --- this is quite impossible since by (2.2)

\[
\dim C_N/IC_N \leq \dim C/I \leq \dim \hat{A}/p < d.
\]

Thus \( A \) is unmixed.

The proof of the last assertion of (1.1) shall be given in the next section, see (3.1).

Remark (2.12). \( \text{Proj} R \) is not necessarily regular even though \( \text{Proj} R \) is normal and depth \( A > 0 \). In fact, provided \( d \geq 2 \) and depth \( A > 0 \), \( \text{Proj} R \) is regular if and only if \( A \) is a regular local ring and \( q = m \) (cf., [2, (4.6)]).

3. Normality of the ring \( R \).

In this section we discuss the normality of the ring \( R = \bigoplus_{n \geq 0} q_n \) and our goal is Theorem (3.1). The following conditions are equivalent.

(1) \( A \) is regular and \( \ell_A(q + m^2/m^2) \geq d - 1 \).

(2) \( A \) is an integral domain and \( q \) is integrally closed.

(3) \( q \) is m-full.

(4) \( R \) is normal.

To begin with we recall the definition of m-full ideals. Let \( I \) be an ideal of \( A \). Then we say that \( I \) is m-full if \( mI : x = I \) for some \( x \in m \). The concept of m-full ideal was introduced by D. Rees [5] and basic properties of such ideals are discussed in [7], a few of which we need to prove (3.1).

Let \( v_A(M) \) denote, for a given finitely generated \( A \)-module \( M \), the number of elements in a minimal system of generators for \( M \).

Proposition (3.2) ([7, Theorem 2 and 3]). Let \( I \) be an m-primary ideal of \( A \) and assume that \( I \) is m-full. Let \( x \in m \) such that \( mI : x = I \). Then

\[
v_A(J) \leq v_A(I) = \ell_A(A/I + xA) + v_A(I + xA/xA)
\]

for any ideal \( J \) of \( A \) containing \( I \).

Let \( I \) be an ideal of \( A \). Then an element \( x \) of \( A \) is called integral over \( I \) if \( x \)
satisfies an equation
\[ x^N + c_1 x^{N-1} + \ldots + c_N = 0 \]
with \( c_i \in R \ (1 \leq i \leq N) \). Recall that \( I \) is said to be integrally closed if every element of \( A \) which is integral over \( I \) belongs to \( I \).

The next result is due to D. Rees and a proof may be found in [7] (cf., Theorem 5).

**Proposition (3.3).** Suppose that \( A \) is an integral domain with infinite residue class field. Then every integrally closed ideal of \( A \) is \( m \)-full.

**Proof of Theorem (3.1).** (4) \( \Rightarrow \) (2) Let \( N = mR + R_+ \) and \( P \in \text{Ass } R \). Then \( P \subseteq N \), as \( P \) is graded and as \( N \) is a unique graded maximal ideal of \( R \). As \( R_N \) is normal, it is an integral domain and so \( PR_N = (0) \), whence \( P = (0) \). Thus \( R \) is an integral domain and so \( A \) is. Let us identify \( R \) with the \( A \)-subalgebra \( A[cT, c \in q] \) of \( A[T] \) where \( T \) is an indeterminate over \( A \). Let \( c \in A \) which is integral over \( q \). Then as \( cT \) is integral over \( R \), we get \( cT \in R \); hence \( cT \in qT \); that is \( c \in q \). Thus \( q \) is integrally closed.

(3) \( \Rightarrow \) (1) Take \( x \in m \) so that \( m^a : x = q \). Then by (3.2) we find that
\[
\ell_A(m) \leq \nu_A(q) = \ell_A(A/q + xA) + \nu_A(q + xA/xA).
\]
So \( A \) is a regular local ring, since \( \nu_A(m) \leq \nu_A(q) = d \). Furthermore we get \( \ell_A(A/q + xA) = 1 \), because \( \ell_A(A/q + xA) \geq 1 \) and \( \nu_A(q + xA/xA) \geq d - 1 \). Thus \( q + xA = m \), that is \( \ell_A(q + m^2/m^2) \geq d - 1 \).

(2) \( \Rightarrow \) (1) Passing to the ring \( A[U]_{m, A[U]} \) where \( U \) is an indeterminate over \( A \), we may assume that the field \( A/m \) is infinite. Then as \( q \) is \( m \)-full by (3.3), our implication follows from \( (3) \Rightarrow (1) \).

(1) \( \Rightarrow \) (3) Let \( x \in m \) with \( m = (a_1, \ldots, a_i, \ldots, a_d, x) \) for some \( 1 \leq i \leq d \). Then we get
\[
\ell_A(A/q + xA) + \nu_A(q + xA/xA) = \ell_A(A/m) + \nu_A(m/xA),
\]
Recalling the exact sequence
\[
0 \rightarrow m^a : x/m^a \rightarrow A/m^a \rightarrow A/mq + xA \rightarrow 0
\]
of \( A \)-modules, we have
\[
\ell_A(m^a : x/m^a) = \ell_A(A/mq + xA). \tag{b}
\]
Notice that
\[
\ell_A(A/q + xA) + \nu_A(q + xA/xA) = \ell_A(A/q + xA) + \ell_A(q + xA/mq + xA)
= \ell_A(A/mq + xA). \tag{c}
\]
Then we get by (a) and (b) that
\[
\ell_A(m^a : x/m^a) = \ell_A(q/m^a), \tag{c}
\]
as \( \ell_A(q/m^a) = \nu_A(q) = d \). Since \( m^a : x \supseteq q \), it follows from (c) that \( m^a : x = q \). Thus \( q \)
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is n-full.

(1) \implies (4) Let \( N = \mathfrak{m}R + R_+ \). Then as Proj \( R \) is normal (cf. (2.6)), we get that the local ring \( R_P \) is normal for any prime ideal \( P (P \neq N) \) of \( R \). On the other hand as \( R \) is a Cohen-Macaulay ring (cf. [1]), we see depth \( R_N = \dim R_N = d + 1 \geq 2 \); so the local ring \( R_N \) must be normal too. This completes the proof of (3.1).

References

5. D. Rees, Lectures at Nagoya University, 1983.
7. J. Watanabe, \( m \)-full ideals, in preprint.

Shiro GOTO
DEPARTMENT OF MATHEMATICS
NIHON UNIVERSITY

Kikumichi YAMAGISHI
DEPARTMENT OF MATHEMATICS
SCIENCE UNIVERSITY OF TOKYO