On Turing Machine with Multi-Heads

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§ 1. Introduction

In the past days, a Turing machine that consists of a two-way infinite tape with a head, had been considered. On the other hand, in these days, a Turing machine that consists of a one-way infinite tape and finitely many two-way infinite tapes with their respective heads, has been considered. Of course, the latter machine has also been considered, from the old days. But, it has been well-known that the former and the latter are equivalent in the following sense: a function computed by the one is computed by the other, too. And the former is simpler than the latter in their structures. So, the former machine had been treated in the past.

Nevertheless, there is a difference between the former and the latter, in the time required for computing a function. Usually, the former is slower than the latter. And the time required for computing has become a center of interest recently. So, the latter machine has been treated in these days.

By the way, although each tape of these machines has a single head, our eyes can see many objects simultaneously. So, I believe that it is significant to deal with a Turing machine with multi-heads.

§ 2. One-tape Turing machines vs. many-tape Turing machines

Definition 2.1 Let $Q$ be a finite set of states. Let $\Sigma$ be an alphabet (i.e. a finite set of symbols). Let $B \in \Sigma$ be a blank. Let $q_0$, $q_A$ and $q_R$ in $Q$ be initial, accepting and rejecting states, respectively. Let $\delta : Q \times \Sigma \rightarrow Q \times \Sigma \times \{L, C, R\}$ be a many-valued transition function, where $L$, $C$ and $R$ denote “left”, “center” and “right”, respectively. Then we say that $M=\langle Q, \Sigma, B, \delta, q_0, q_A, q_R \rangle$ is a non-deterministic one-tape Turing machine. Especially, when $\delta$ is a one-valued function, we say that $M$ is a deterministic one-tape Turing machine. (Cf. Figure 1.) We abbreviate these as 1-(N) DTM.

Definition 2.2 Let $M$ be the same as Definition 2.1 except $\delta$. Let $\delta : Q \times \Sigma^{m+1} \rightarrow Q \times \Sigma^m \times \{C, R\} \times \{L, C, R\}^m$ be a many-valued function, where $m>0$ is a fixed integer. Then we say that $M=\langle Q, \Sigma, B, \delta, q_0, q_A, q_R \rangle$ is a non-deterministic $m+1$-tape Turing machine.
machine. Especially, when \( \delta \) is a one-valued function, we say that \( M \) is a deterministic \( m+1 \)-tape Turing machine. (Cf. Figure 2.) We abbreviate these as \( m+1-(N)DTM \).

The uppermost one-way infinite tape in Figure 2 is a read-only tape for input. And the other two-way infinite tapes are read-write tapes for working.

**Lemma 2.3** Suppose that an \( m+1-(N)DTM \) \( M = \langle Q, \Sigma, B, \delta, q_0, q_A, q_R \rangle \) is given. Then, there is a \( 1-(N)DTM \) \( M' = \langle Q', \Sigma', B, \delta', q_0, q_A, q_R \rangle \) such that there is a computation in \( M \) starting with input \( \alpha \) at state \( q_0 \) and ending at state \( q_A \) (or \( q_R \)) in time \( T(|\alpha|) \), if and only if there is a computation in \( M' \) starting with input \( \alpha \) at state \( q_0 \) and ending at state \( q_A \) (or \( q_R \)) within time \( 2(m+1)|\alpha|^2 + 2(m+2)^2(T(|\alpha|)^2 + 3T(|\alpha|) + 2) \), where \( \alpha \) is in \( (\Sigma - \{B\})^* \), \( |\alpha| \) is the length of \( \alpha \), and \( T \) is a number-theoretic function. (\( \Sigma^* \) is the set of all finite strings consisting of the symbols in \( \Sigma \) that is called as Kleene star.)

**Proof** Let \( \Sigma' = \bigcup_{a_1 \in \Sigma} \{a, a^i, a^h, a^{ih}\} \), where \( a^i \), \( a^h \) and \( a^{ih} \) are new symbols corresponding to \( a \) and different from each other. We make \( M' \) which simulates the move of \( M \), as following.

First of all, when the initial \( \alpha \) is of \( a_1a_2\cdots a_n \), we want to change it to
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\[a_1 b_1 a_2 b_2 \cdots a_m b_m a_{m+1} b_{m+1} \cdots b_n a_n,\]
where superscripts \(l\) and \(h\) of \(a\) in \(M'\) denote that the corresponding positions are leftmost and heading respectively. (When \(|\alpha| = 0\), we may assume \(T(|\alpha|) = 0\).) \(M'\) spends at most \(2(m+1)n\) steps to separate the adjoining symbols \(a_i\) and \(a_{i+1}\) (\(1 \leq i < n\)) in \(\alpha\) exactly \(m\) squares and to return back the leftmost position if we choose \(Q'\) and \(\delta'\) appropriately, where \(n = |\alpha|\). Therefore, \(\alpha\) changes to \(a_1 b_1 a_2 b_2 \cdots a_m b_m a_{m+1} b_{m+1} \cdots b_n a_n\) after at most \(2(m+1)|\alpha|^2\) steps.

Secondly, we note the move of \(M\) from step \(t\) to step \(t+1\). Each tape of \(M\) has scanned at most \(t\) consecutive squares. Therefore, the head of each tape of \(M\) is on the square which is at most \(2t\) squares right from the leftmost position of the \(m+1\) tapes. Hence, \(M'\) spends at most \(4t(m+1)\) steps to read all symbols of the heads of \(M\). And, when the head of each tape of \(M\) moves one square right or left by the transition function \(\delta\), \(M'\) spends at most \(4(m+1)(t+1)\) steps to move the corresponding heading position to exactly \(m+1\) squares right or left and to return back the leftmost position if we choose \(Q'\) and \(\delta'\) appropriately. For example, when one of the tapes of \(M\) changes as Figure 3, the corresponding part of the tape of \(M'\) changes as Figure 4.

Since \(M\) has \(m+1\) tapes, one step of \(M\) from \(t\) to \(t+1\) corresponds to at most \(4t(m+1) + 4(m+1)^2(t+1) < 4(m+2)^2(t+1)\) steps of \(M'\). Therefore, the computation in \(M'\) reaches to \(q_A\) (or \(q_R\)) within time \(2(m+1)|\alpha|^2 + \sum_{t=0}^{T(|\alpha|)} 4(m+2)^2(t+1) - 2(m+1)|\alpha|^2 + 2(m+2)^2(T(|\alpha|)^2 + 3T(|\alpha|) + 2).

**Lemma 2.4** Suppose that a \(1\text{-}(N)\) DTM \(M = \langle Q, \Sigma, B, \delta, q_0, q_A, q_R \rangle\) is given. Then there is an \(m+1\text{-}(N)\) DTM \(M' = \langle Q', \Sigma, B, \delta', q_0, q_A, q_R \rangle\) such that there is a computation in \(M\) starting with input \(\alpha\) at \(q_0\) and ending at \(q_A\) (or \(q_R\)) in time \(T(|\alpha|)\), if and only if there is a computation in \(M'\) starting with input \(\alpha\) at \(q_0\) and ending at \(q_A\) (or \(q_R\)) within time \(2|\alpha| + T(|\alpha|)\).

**Proof** First, \(M'\) copies input \(\alpha\) onto one of the working tapes and returns back the leftmost position at \(2|\alpha|\) steps, and then, \(M'\) simulates \(M\) at the same steps as \(M\), if we choose \(Q'\) and \(\delta'\) appropriately.

**Definition 2.5** A subset of \((\Sigma - \{B\})^*\) is called as a language. When \(T\) is a number-theoretic function, we say that a language \(L\) is recognized by Turing machine \(M\) in time \(T\) if the following condition holds:

1. if \(\alpha \in L\) then the computation in \(M\) starting with \(\alpha\) at state \(q_0\) ends at state \(q_A\) within time \(T(|\alpha|)\),
(2) if $\alpha \in L$ then the computation in $M$ starting with $\alpha$ at state $q_j$ ends at state $q_R$ within time $T(|\alpha|)$.

**Definition 2.6** We define the sets of the languages $P_0$, $NP_0$, $P$ and $NP$, as follows:

- $P_0 = \{L |$ there are a 1-DTM $M$ and a polynomial function $T$ such that $L$ is recognized by $M$ in time $T\}$,
- $NP_0 = \{L |$ there are a 1-NDTM $M$ and a polynomial function $T$ such that $L$ is recognized by $M$ in time $T\}$,
- $P = \{L |$ there are an $m+1$-DTM $M$ ($m > 0$) and a polynomial function $T$ such that $L$ is recognized by $M$ in time $T\}$,
- $NP = \{L |$ there are an $m+1$-NDTM $M$ ($m > 0$) and a polynomial function $T$ such that $L$ is recognized by $M$ in time $T\}$.

**Theorem 2.7** $P_0 = NP_0$ if and only if $P = NP$.

**Proof** Trivial from Lemmas 2.3 and 2.4.

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§ 3. Turing machines with single-head vs. Turing machines with multi-heads

**Definition 3.1** Let $M$ be the same as Definition 2.2 except $\delta$. Let $\delta : Q \times \Sigma^{(m+1)n} \rightarrow Q \times \Sigma^m \times \{C, R\}^n \times \{L, C, R\}^m$ be a many-valued transition function, where $m$ and $n$ are fixed positive integers. Then we say that $M = \langle Q, \Sigma, B, \delta, q_0, q_A, q_R \rangle$ is a non-deterministic $m+1$-tape Turing machine with $n$-heads. Especially, when $\delta$ is a one-valued function, we say that $M$ is a deterministic $m+1$-tape Turing machine with $n$-heads. (Cf. Figure 5.) We abbreviate these as $(m+1, n)$-DTM.

![Figure 5](image-url)
Before stating the following Lemma 3.2, we explain the move of \((m+1, n)-(N)\) DTM. Let \(h_1^0, \ldots, h_n^0\) be the heads on the input tape. Let \(h_1^1, \ldots, h_n^1\) be the heads on the \(i\)-th working tape. In principle, the symbol in the square scanned by each \(h_j^i\) \((1 \leq i \leq m, 1 \leq j \leq n)\) is rewritten and \(h_j^i\) is moved by \(\delta\) according to the combination of the state and all scanned symbols in the moment. However, there is a possibility that heads more than one are on the same square. In this case, it follows the rule that the symbol in the square scanned by them changes on the basis of \(h_j^i\) with the minimum \(j\).

For example, let \(m=1, n=2\) and \(\delta(q, a, a, b, b) = \langle q', a, b, c, R, L, R \rangle\), where the arrangement of the variables of \(\delta\) is in the order of \(h_1^0, \ldots, h_n^0, h_1^1, \ldots, h_n^1, \ldots, h_m^m\). When all heading positions on the working tape are different from each other, it moves as in Figure 6. But when some of them coincide with each other, it moves as in Figure 7.

**Lemma 3.2** Suppose that an \((m+1, n)-(N)\) DTM \(M = \langle Q, \Sigma, B, \delta, q_0, q_A, q_R \rangle\) is given. Then there is an \((m+1)n+1-(N)\) DTM \(M' = \langle Q', \Sigma', B, \delta', q_0, q_A, q_R \rangle\) such that there is a computation in \(M\) starting with input \(\alpha\) at \(q_0\) and ending at state \(q_A\) (or \(q_R\)) in time \(T(|\alpha|)\), if and only if there is a computation in \(M'\) starting with input \(\alpha\) at \(q_0\).
and ending at \( q_A \) (or \( q_R \)) within time \( 4T(|\alpha|)^2 + 4(n+1)T(|\alpha|) + 2(|\alpha| + n) \).

**Proof** First of all, let \( \Sigma' \) be the same as the proof of Lemma 2.3. Now, \( M' \) copies input \( \alpha \) onto the first \( n \) tapes of the working ones and carries back the heads of working tapes of \( M' \) to their leftmost positions as Figure 8, within at most \( 2(|\alpha| + n) \) steps, if we choose \( Q' \) and \( \delta' \) appropriately. And then, the symbol \( \alpha_i \) is changed \( \alpha_i' \) and the symbol \( \alpha_l \) on each heading position of \( M \) is changed to \( \alpha_l' \).

![Figure 8](image)

Secondly, we consider the move of \( \delta \) from time \( t \) to \( t+1 \). Each tape of \( M \) has scanned at most \( n+2t \) consecutive squares. Therefore, \( M' \) spends at most \( 2(n+2t) \) steps to read all symbols of the heads of \( M \), and at most \( 2(n+2t+2) \) steps to simulate the action of one step of \( M \) and to return back their leftmost position, if we choose \( Q' \) and \( \delta' \) appropriately. For example, when \( M \) is moved by \( \delta \) as Figure 9, \( M' \) is moved by \( \delta' \) as Figure 10.

![Figure 9](image)

Hence, when \( M \) spends \( T(|\alpha|) \) steps to start input \( \alpha \) at \( q_0 \) and to halt at \( q_A \) or \( q_R \), \( M' \) spends at most \( 2(|\alpha| + n) + \sum_{t=0}^{T(|\alpha|)-1} 4(n + 2t + 2) = 4T(|\alpha|)^2 + 4(n + 1)T(|\alpha|) + 2(|\alpha| + n) \) steps to start input \( \alpha \) at \( q_0 \) and to halt at \( q_A \) or \( q_R \).

**Definition 3.3** We define the sets of the languages \( P_h \) and \( NP_h \) as follows:

\[ P_h = \{ L \mid \text{there are an } (m+1,n) \cdot \text{DTM } M \text{ and } (m, n > 0) \text{ and a polynomial function } T \]
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Theorem 3.4 \( P = NP \) if and only if \( P^h = NP^h \).

Proof Trivial from Lemma 3.2.

§ 4. Conclusion

It is trivial that \( P \subseteq NP \) holds absolutely, because that DTM is a special case of NDTM. But we cannot know whether \( P = NP \) holds absolutely, or not. So, necessary and sufficient conditions to be \( P = NP \) are studied by many mathematicians. We also studied a few works about it. I think that \( P = NP \) has a relation to the “axiom of choice” in set theory. I would like to study about it from now, and to get better results.

Remark Consider a Turing machine that consists of a one-way infinite read-only tape for input and infinitely many two-way infinite read-write tapes for working. But, consider that this machine can scan squares of only finite tapes each time. Let \( M = \langle Q, \Sigma, \delta, q_0, q_A, q_R \rangle \) be the same as Definition 2.2 except \( \delta \). Let \( \delta : Q \times \Sigma^+ \to Q \times \Sigma^* \times [C, R] \times [L, C, R]^* \) be a many-valued transition function, where \( \Sigma^+ \) is the set of all non null strings of \( \Sigma \), i.e. \( \Sigma^+ = \Sigma \Sigma^* \). Then, we call \( M \) as a non-deterministic \( \infty \)-tape Turing machine. Especially, when \( \delta \) is one-valued, we call \( M \) as a deterministic \( \infty \)-tape Turing machine. We abbreviate these as \( \infty \)-DTM. Assume that the number of used tapes of an \( \infty \)-DTM \( M \) augments less than some constant number at each time. When there is a computation of \( M \) halting within a polynomial
time of $|\alpha|$, we cannot reduce this to an $m+1$-(N)DTM $M'$ with a fixed $m>0$ halting within a polynomial time of $|\alpha|$. Hence, when we write the corresponding sets of $P$ and $NP$ as $P_\infty$ and $NP_\infty$ respectively, we cannot insist the equivalence between $P = NP$ and $P_\infty = NP_\infty$. I think that it perhaps fails.

References